# 2 ERRORS IN SURVEYING

## 2.1 INTRODUCTION

To understand the subject of errors in surveying, let us assume that we need to measure a certain distance AB using a measuring tape. To do so, we gave a team of two people a tape and asked them to measure the distance under the following circumstances:

- Perform the measurement on a beautiful day (no wind, comfortable temperature and so on) and write down the result.
- Repeat the measurement immediately on the same day with the same tape.
- Repeat the measurement on a windy day.
- Repeat the measurement on a hot day.
- Repeat the measurement on a cold day.
- Repeat the measurement under the previous conditions but with a different tape.
- Repeat the measurement under the previous conditions, but two different people are asked this time to do the job.

If we compare the various measurements obtained in the different previous scenarios, we will notice that they are not equal. The reason is that all surveying operations are subject to three sources of error, which could lead to harmful and unexpectable results. These sources are:

a) The imperfections of the instruments,
b) The fallibility of the human operator, and
c) The uncontrollable nature of the environment.

Actually, no surveying measurement is exact and free of error (unless by chance), and the true values of the measured parameters are never known. Therefore, a surveyor must thoroughly understand the sources of error in the various methods of surveying, as well as, the methodology for evaluating the achievable accuracy of a surveying program.

Due to the high importance of the subject of errors, and the need to know how to control, avoid and minimize them, I felt the need to introduce this subject to the reader before he/she learns about the techniques and equipment needed to perform the various surveying operations. This chapter will present the fundamental principles of measurement errors and the basic statistical techniques used for evaluating the accuracy of various methods of surveying and of survey results.

## 2.2 ERRORS IN SURVEYING MEASUREMENTS

The *true error* in a surveying measurement is defined as the difference between the measured value of a parameter and its true value.

Let $e_i$ = true error
$x_i$ = measured value
$x$ = true value

$$\Rightarrow e_i = x_i - x \qquad \text{.......................(2.1)}$$

But since, as mentioned earlier, the true value ($x$) can never be determined, the true error ($e_i$) too can never be exactly determined. Therefore, the error in a measurement must be estimated or calculated by comparing it with another more accurately determined value of the same parameter, such as the mean of several measurements. Let $\hat{x}$ represent such a value. Then, an estimate ($v_i$) of the true error ($e_i$) is:

$$v_i = x_i - \hat{x} \quad \text{. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (2.2)}$$

This estimate error is sometimes called the residual error. Once this error is calculated, it should be removed from the measured value ($x_i$) by subtracting it from $x_i$. An alternative way is to add a correction ($C_i$) to the measured value. This correction is equal to the error in magnitude but with opposite sign, that is: correction = - error.

In general, errors in surveying measurements can be divided into three different types:

1. Blunders (also referred to as mistakes),
2. Systematic errors, and
3. Random errors (also referred to as compensating or accidental errors).

### 2.2.1 BLUNDERS (MISTAKES)

These are simply mistakes caused by human carelessness, fatigue and haste. Blunders can be positive or negative, large or small and their occurrence is unpredictable. Some examples of blunders are the transposition of digits in recording a measurement (such as recording 43.18 instead of 34.18) and sighting a wrong target when measuring an angle.

Blunders are disastrous if left in the surveying measurements, and therefore, must be eliminated by careful work and by using field procedures that provide checks for blunders as will be explained later in several places in this book.

### 2.2.2 SYSTEMATIC ERRORS

These are mostly caused by the maladjustment of the surveying instruments and by the uncontrollable nature of the environment. Both the signs and magnitudes of systematic errors behave according to a particular system or physical law of nature, which may or may not be known. When the law of occurrence is known, systematic errors can be calculated and eliminated from the measurements.

One example of systematic errors is the tape length error when, for various reasons, the actual length of the tape will be different from its nominal length under calibration conditions.

**EXAMPLE 2.1:**

A line was found to be 376.4 m when measured with a tape, which is believed to be 20 m long (nominal length). On checking, the actual tape length was found to be 20.04 m. What is the correct length of the line?

**SOLUTION:**

$$\text{Correct length of the line} = \text{measured length} \cdot \frac{\text{actual tape length}}{\text{nominal tape length}}$$

$$\Rightarrow \text{Correct length of the line} = 376.4 \text{ x } \frac{20.04}{20} = 377.2 \text{ m}$$

For areas (see Chapter 3: section 3.7):

$$\text{Correct Area} = \text{measured area x } \left( \frac{\text{actual length of the tape}}{\text{nominal length of the tape}} \right)^2$$

A special type of systematic error is an error that always occurs with the same sign and magnitude and is therefore often referred to as a *constant error*. The most common source of constant error is the measuring instruments. For example a 30-m tape may in fact be missing the first 0.10 m (i.e., 10 cm) due to the deterioration of the tape after the repeated use. Then, if not noticed, every time the tape is used would contain a constant error of +0.10 m. Constant errors of this type can be detected by careful attention and calibration of the instruments. More examples of systematic errors will be given in the next chapters.

### 2.2.3 RANDOM ERRORS (COMPENSATING OR ACCIDENTAL ERRORS)

These are caused by imperfections of the measuring instruments, imperfections of the surveyor to make an exact measurement, and the uncontrollable variations in the environment. These errors can be minimized by using better instruments and properly designed field procedures and by making repeated measurements.

Random errors have the following characteristics:

1. Positive and negative errors of the same magnitude occur with the same frequency.
2. Small errors occur more frequently than large ones.
3. Very large errors seldom occur.
4. The mean of an infinite number of observations is the true value.

For example, let us assume that a distance is measured using the same instrument and the same degree of care, a large number of times, say 1000 times. Then, the mean or average of the 1000 repeated measurements is computed, and the estimated (residual) error in each individual length measurement is calculated using Equation (2.2). The estimated error computed in this manner is called the deviation from the mean because it is a measure of how far is the measurement from the mean. Now, calculate the range of these errors (range = maximum error – minimum error), divide this range into a suitable number (5 to 8 intervals) of equal intervals and count the number of occurrences in each interval. Plotting the frequency of occurrence against the interval limits of the estimated error may result in a histogram similar to that shown in Figure 2.1.

For an infinite number of repetitions of the measurements, this histogram approximates to a continuous normal curve with the following probability density function (p.d.f):

$$f(v) = \frac{1}{\sigma\sqrt{2\pi}}.e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^2} \qquad \text{(2.3)}$$

Where v = random error, and

σ = Standard error or deviation of the measurements (see next section).

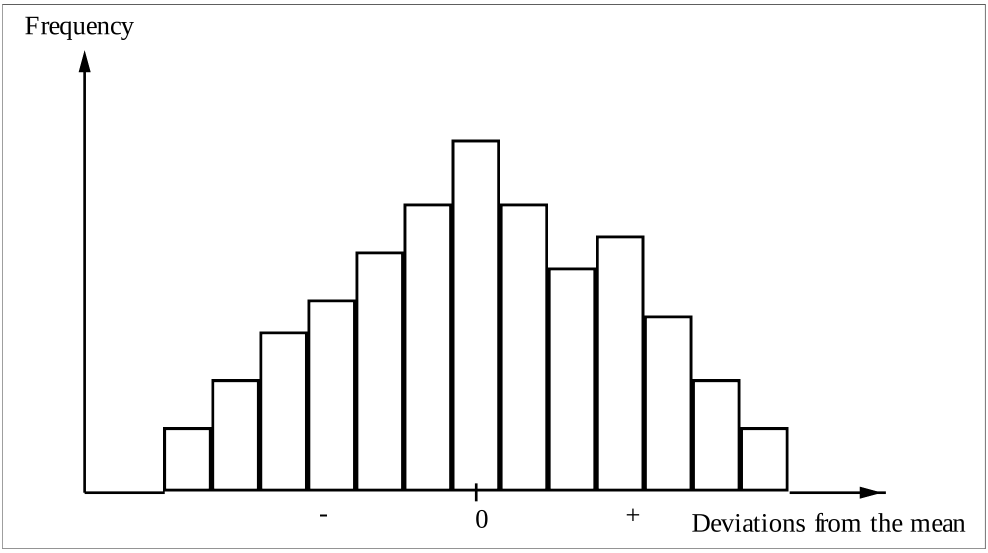


**FIGURE 2.1:** A histogram which shows the distribution of random errors.

This continuous curve is shown in Figure 2.2.

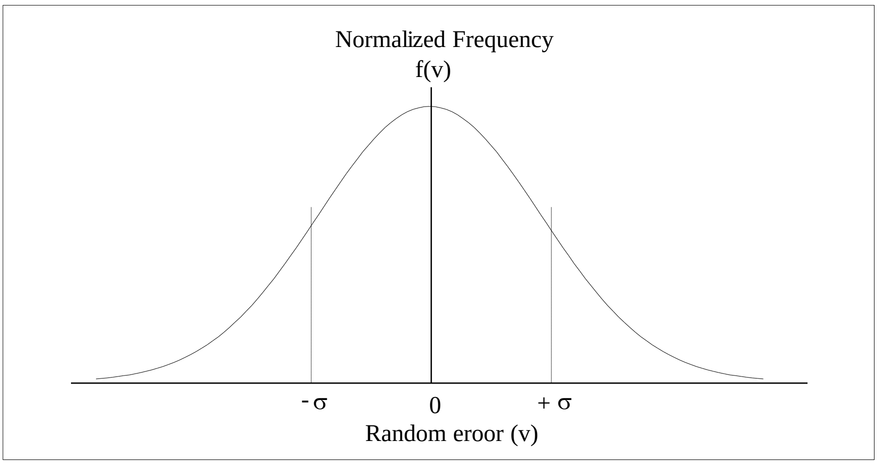


**FIGURE 2.2:** Normal curve of error.

The normal curve is symmetrical about v = 0. The probability that the random error in a measurement takes on a value between a and b, is equal to the area under the curve and bounded by the values of a and b as shown in Figure 2.3

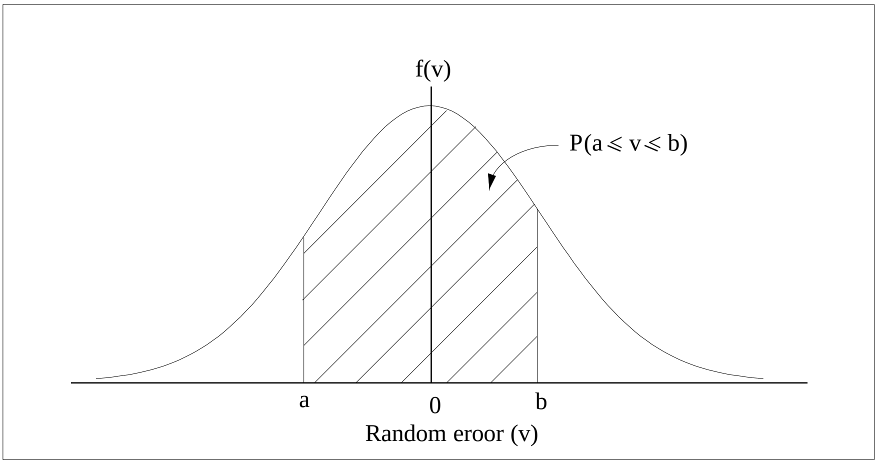


**FIGURE 2.3:** Probability of random errors.

In mathematical terms, if P(a ≤ v ≤ b) represents that probability, then

$$P(a \leq v \leq b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} . e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^2} . dv \qquad \ldots\ldots\ldots\ldots(2.4)$$

The curve is normalized so that the area under the entire curve is equal to 1. Since this integral is so complicated, probability values can be taken from already prepared tables, which can be found in most statistics book.

Some representative probabilities for selected error ranges are:

| Error Range | Probability (%) |
|---|---|
| $\pm$ 0.6745 $\sigma$ | 50.0 |
| $\pm$ 1.00 $\sigma$ | 68.3 |
| $\pm$ 1.6449 $\sigma$ | 90.0 |
| $\pm$ 2.00 $\sigma$ | 95.4 |
| $\pm$ 3.00 $\sigma$ | 99.7 |

## 2.3 MEAN, STANDARD DEVIATION AND STANDARD ERROR OF THE MEAN

Let $x_1, x_2, x_3 \ldots x_n$ be n repeated measurements of the same quantity, and let us assume that all these measurements were made with the same instrument and same degree of care. Then:

1) The mean denoted by $\overline{x}$, of the n measurements is computed as follows:

$$\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n} \qquad \ldots (2.5)$$

2) An estimate of the standard error $\hat{\sigma}_x$ of *one measurement* of the quantity is:

$$\hat{\sigma}_x = \pm \sqrt{\frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n - 1}} \qquad \ldots (2.6)$$

The standard error is sometimes called the standard deviation or root-mean-square (RMS) error of a single measurement. It is a measure for the error in a single measurement as compared to the calculated mean.

3) An estimate of the standard error of the *mean of the n measurements*, to be denoted by $\hat{\sigma}_{\overline{x}}$, can be computed as follows:

$$\hat{\sigma}_{\overline{x}} = \pm \sqrt{\frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n(n - 1)}} \qquad \ldots (2.7)$$

Or

$$\hat{\sigma}_{\overline{x}} = \frac{\hat{\sigma}_x}{\sqrt{n}} \qquad \ldots (2.8)$$

$\hat{\sigma}_{\overline{x}}$ is also called the RMS error of the mean. It is a measure for the error in the mean itself as compared to the true value or an acceptable estimate of it.

## 2.4 PROBABLE AND MAXIMUM ERRORS

The *probable error* of a measurement is defined to be equal to 0.6745σ. There is a 50% probability that the actual error exceeds the probable error, as well as, a 50% probability that it is less than the probable error. The probable error was widely used in surveying in the past as a measure of precision, now it is replaced by the standard error.

The *maximum error* in a measurement is defined as being equal to 3σ. There is a 99.7% probability that the actual error falls within 3σ, and only a 0.3% probability that the actual error exceeds 3σ.

**Example**: If the standard error of an angle measurement is ± 3.0 seconds, then,
The probable error = ± (0.6745 x 3.0) = ± 2.0 seconds
The maximum error = ± (3 x 3.0) = ± 9.0 seconds

The maximum error is usually used as a measure for detecting and isolating blunders from the surveying measurements. For example, after the mean and standard deviation of *n* repeated measurements have been computed, the deviation ($v_i$) of each measurement from the mean can be computed ($v_i = x_i - \bar{x}$). If any measurement deviates from the mean by more than 3σ, the measurement is considered to have a blunder. It is rejected, and a new mean and standard deviation are computed without this particular measurement.

**EXAMPLE 2.2:**

A distance was repeatedly measured 12 times, and the following results (in meters) were recorded:
58.78, 58.83, 58.80, 58.85, 58.18, 58.77, 58.79, 58.80, 58.81, 58.82, 58.79 & 58.82

Check these measurements for the existence of any blunders, reject them (if any), and compute the mean, the standard deviation, and estimated standard error of the mean.

**SOLUTION:**

| Measurement (m) | First Iteration $v_i = d_i - \bar{d}$ (m) | Second Iteration $v_i = d_i - \bar{d}$ (m) |
|---|---|---|
| 58.78 | 0.03 | -0.03 |
| 58.83 | 0.08 | 0.02 |
| 58.80 | 0.05 | -0.01 |
| 58.85 | 0.10 | 0.04 |
| 58.18 | -0.57 | blunder $\Rightarrow$ rejected |
| 58.77 | 0.02 | -0.04 |
| 58.79 | 0.04 | -0.02 |
| 58.80 | 0.05 | -0.01 |
| 58.81 | 0.06 | 0.00 |
| 58.82 | 0.07 | 0.01 |
| 58.79 | 0.04 | -0.02 |
| 58.82 | 0.07 | 0.01 |

<u>First Iteration</u>: (n = 12)

Mean = 58.75 m

Standard deviation = ± 0.18 m

Estimated standard error of the mean = $\frac{\pm 0.18}{\sqrt{12}}$ = ± 0.05 m

Maximum error of a single measurement = ± 3 x 0.18 = ± 0.54 m

$\Rightarrow$ Reject measurement 58.18 m (possibly the surveyor recorded 58.18 m instead of 58.81 m)

<u>Second Iteration</u>: (n = 11)

Mean = 58.81 m

Standard deviation = ± 0.02 m

Estimated standard error of the mean = $\frac{\pm 0.02}{\sqrt{11}}$ = ± 0.007 m ≈ ± 0.01 m

Maximum error of a single measurement = ± 3 x 0.02 = ± 0.06 m

$\Rightarrow$ No more measurements are rejected.

## 2.5 PRECISION AND ACCURACY

***Precision:*** A measurement is considered to have high precision if it has a small standard deviation. For example, assume that team A measured a distance with $\hat{\sigma}_A = \pm 0.05$ m, and team B measured the same distance with $\hat{\sigma}_B = \pm 0.10$ m. The measurement of team A is said to be more precise than that of team B. Figure 2.4 shows that a large standard deviation means a flatter distribution curve for the random errors.

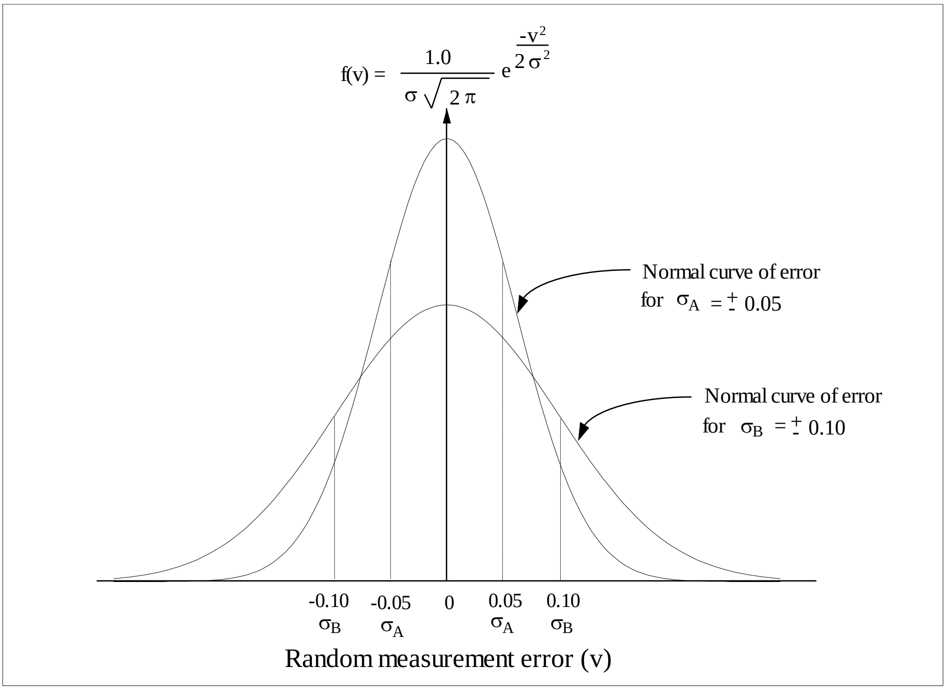


**FIGURE 2.4:** Standard error and the distribution of random errors.

***Accuracy:*** A measurement is considered to have high accuracy if it is close to the true value. High precision does not necessarily mean high accuracy. A measurement that is highly precise is also highly accurate if it contains little or no systematic errors with all blunders removed. Figure 2.5 shows the four possible combinations of precision and accuracy.

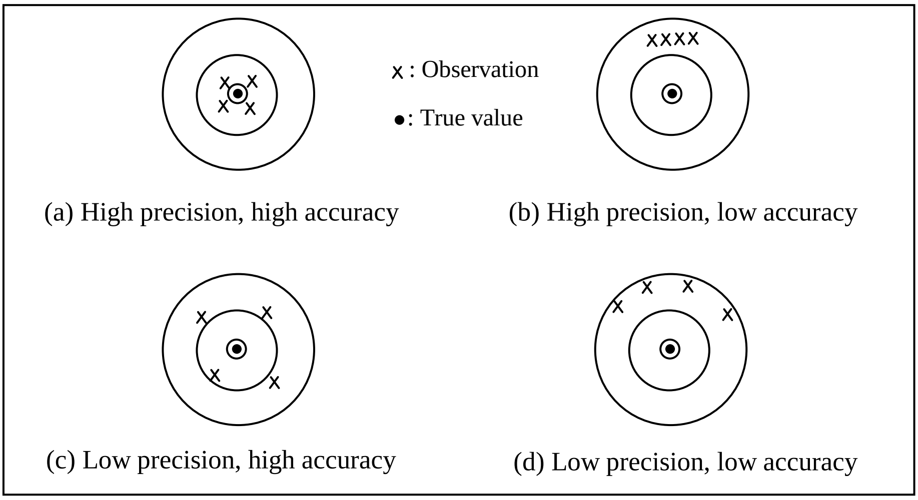


**FIGURE 2.5:** Possible combinations of precision and accuracy.

**EXAMPLE 2.3:**

A distance was measured by two independent parties, with the following results:

Party A: $D_A = 257.361 \pm 0.032$ m

Party B: $D_B = 257.538 \pm 0.011$ m

This distance was later measured by highly calibrated EDM (see Chapter 6) and found to be 257.407 m (with all blunders and systematic errors removed).

Compare between the two teams in terms of precision and accuracy.

**SOLUTION:**

$\hat{\sigma}_B = \pm\, 0.011 \text{ m} < \hat{\sigma}_A = \pm\, 0.032 \text{ m}$

$\Rightarrow$ Party B measurement is more precise than Party A.

True value of the distance can be assumed to be D = 257.407 m

Absolute error in party A measurement = $|257.361 - 257.407| = 0.046\,\text{m}$

Absolute error in party B measurement = $|257.538 - 257.407| = 0.131\,\text{m}$

$0.131 > 0.046 \Rightarrow$ Party B measurement is less accurate than Party A.

**Result:** Party B measurement is more precise but less accurate than Party A measurement.

In general, to obtain high precision and high accuracy in surveying, the following strategies must be followed:

1. Follow techniques that will help detect and eliminate all the blunders.
2. Eliminate or correct all systematic errors by frequent calibration and adjustment of the instruments, and
3. Minimize the random errors by using good instruments and field procedures.

## 2.6 RELATIVE PRECISION

Relative precision is a term that is commonly used to describe the precision of *distance measurement* in surveying. Suppose that a distance D is measured with a standard error $\sigma_D$, then:

Relative precision of the measured distance at 1 σ = $\dfrac{1}{D/\sigma_D}$ . . . . . . . . . . . . (2.9)

Usually, it is adequate enough to round off the denominator in the relative precision fraction to one or two non-zero digits. For example: a distance (D) was measured and found to be 4576.2 ± 0.3m, then:

Relative precision of the measured distance at 1 σ = $\dfrac{1}{\frac{4576.2}{0.3}} = \dfrac{1}{15{,}254} \approx \dfrac{1}{15{,}000}$

This means that if we measure a distance of length 15,000 units (could be m or ft), then there is a 68.3% chance that the error is within 1 unit. Some agencies choose to represent the relative precision at 2 or 3σ level accuracy. For the previous distance:

Relative precision of measured distance at 3 σ = $\frac{1}{\frac{4576.2}{3 \times 0.3}} = \frac{1}{5,085} \approx \frac{1}{5,000}$

This means that if we measure a distance of length 5,000 units, then there is a 99.7% chance that the error is within 1 unit.

## 2.7 REPEATED MEASUREMENTS

Equation (2.8) [ $\hat{\sigma}_{\bar{x}} = \frac{\hat{\sigma}_x}{\sqrt{n}}$ ] shows that $\hat{\sigma}_{\bar{x}} \propto \frac{1}{\sqrt{n}}$. This means that as the number of repeated observations (*n*) of a parameter increases, the standard error of the mean of these measurements ($\hat{\sigma}_{\bar{x}}$) decreases; leading to a high precision in the measured value of the mean. Equation (2.8) can be modified to look like this:

$$\sqrt{n} = \frac{\hat{\sigma}_x}{\hat{\sigma}_{\bar{x}}}$$

$$\Rightarrow \quad n = \left( \frac{\hat{\sigma}_x}{\hat{\sigma}_{\bar{x}}} \right)^2 \qquad \ldots\ldots (2.10)$$

which means that if the standard deviation of a single measurement is ($\hat{\sigma}_x$), then *n* measurements are needed to achieve a certain value of ($\hat{\sigma}_{\bar{x}}$) for the standard error of the mean.

**For example**: Suppose that an angle can be measured with $\hat{\sigma}_x = \pm 3''$ in one repetition by using a certain instrument, then the number of repetitions required to determine the angle with $\hat{\sigma}_{\bar{x}} = \pm 0.8''$ is:

$$n = \left( \frac{3}{0.8} \right)^2 \approx 14$$

## 2.8 PROPAGATION OF RANDOM ERRORS

So far, the precision and accuracy discussed earlier have been about parameters (such as angles and distances) that are directly measured using surveying equipment. However, it often happens that a quantity is derived from the measured values of other parameters that could be statistically correlated or uncorrelated. For example, a long distance D (Figure 2.6) is obtained by adding the two individually measured sections $d_1$ and $d_2$. Now, assuming that each of these two sections has its own standard deviation (i.e. $\sigma_{d_1}$ & $\sigma_{d_2}$), what would be the standard deviation of the derived quantity D?

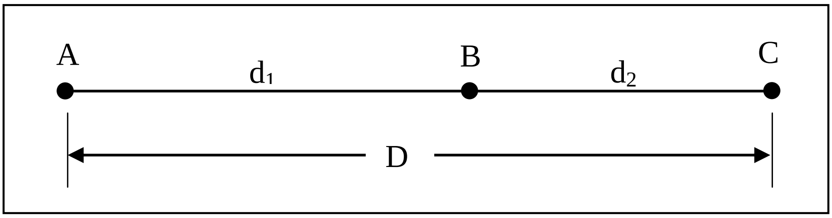


**FIGURE 2.6:** Measuring a long distance in two sections.

The answer for the previous question will be given here for the simplest case when the measured values are statistically uncorrelated. In general, assume that the value of parameter y can be derived from the measured values of *n* other uncorrelated parameters: $x_1, x_2, \ldots x_n$. Let y be related to the *n* parameters by a continuous function:

$$y = F(x_1, x_2, \ldots x_n) \ldots\ldots\ldots\ldots (2.11)$$

Furthermore, let $\hat{\sigma}_{x_i}$ be the estimated standard error of parameter $x_i$ and $\hat{\sigma}_y$ be the estimated standard error of y. Then:

$$\hat{\sigma}_y^2 = \left(\frac{\partial F}{\partial x_1}\right)^2 \hat{\sigma}_{x_1}^2 + \left(\frac{\partial F}{\partial x_2}\right)^2 \hat{\sigma}_{x_2}^2 + \ldots + \left(\frac{\partial F}{\partial x_n}\right)^2 \hat{\sigma}_{x_n}^2 \quad \ldots\ldots\ldots (2.12)$$

This law is called the ***law of propagation of random errors***. It is beyond the scope of this book to give the mathematical derivation for this law, but interested readers can refer to some of the references listed at the end of the book.

From the law of propagation of random errors, it follows that:

1) Error of a sum:
If $y = x_1 + x_2 + \ldots x_n$,
$$\Rightarrow \hat{\sigma}_y = \pm\sqrt{\hat{\sigma}_{x_1}^{\ 2} + \hat{\sigma}_{x_2}^{\ 2} + \ldots + \hat{\sigma}_{x_n}^{\ 2}}$$

2) Error of a product:
If $y = x_1 \cdot x_2$,
$$\Rightarrow \hat{\sigma}_y = \pm\sqrt{x_2^{\ 2} \cdot \hat{\sigma}_{x_1}^{\ 2} + x_1^{\ 2} \cdot \hat{\sigma}_{x_2}^{\ 2}}$$

3) Let y = Ax, where A is a constant and x is a measured quantity
Then,
$$\hat{\sigma}_y = A \, . \, \hat{\sigma}_x$$

**Problem**: Prove the above results.

**EXAMPLE 2.4:**

The radius (r) of a circular tract of land is measured to be 40.25 m with an estimated standard error ($\hat{\sigma}_r$) of ±0.01 m. Compute the area (A) of the tract of land and its estimated standard error ($\hat{\sigma}_A$).

**SOLUTION:**

$$A = \pi r^2 = \pi(40.25)^2 = 5089.58 \text{ m}^2$$

By the law of propagation of random errors:

$$\hat{\sigma}_A^{\ 2} = \left(\frac{\partial A}{\partial r}\right)^2 \hat{\sigma}_r^{\ 2} \Rightarrow \hat{\sigma}_A = \left(\frac{\partial A}{\partial r}\right)\hat{\sigma}_r$$

$$\hat{\sigma}_A = (2\pi r)\,\hat{\sigma}_r = \pm(2\pi \text{ x } 40.25)(0.01) = \pm 2.53 \text{ m}^2$$

$$\Rightarrow A = 5089.58 \pm 2.53 \text{ m}^2$$

**EXAMPLE 2.5:**

The radius (R) of the base of a cone is measured as 14.000 ± 0.002 cm. The height (h) of the cone is measured as 35.000 ± 0.018 cm. What is the standard error of the volume?

**SOLUTION:**

$$V = \frac{\pi R^2 h}{3} = \frac{\pi \text{ x } 14^2 \text{ x } 35}{3} = 7183.775 \text{ cm}^3$$

$\hat{\sigma}_R = \pm 0.002$ cm, $\hat{\sigma}_h = \pm 0.018$ cm

$$\frac{\partial V}{\partial R} = \frac{2\pi Rh}{3} = \frac{2\pi \text{ x } 14 \text{ x } 35}{3} = 1026.254 \text{ cm}^3/\text{cm}$$

$$\frac{\partial V}{\partial h} = \frac{\pi R^2}{3} = \frac{\pi \text{ x } 14^2}{3} = 205.251 \text{ cm}^3/\text{cm}$$

$$\hat{\sigma}_V = \pm\sqrt{\left(\frac{\partial V}{\partial R}\right)^2 \hat{\sigma}_R^{\ 2} + \left(\frac{\partial V}{\partial h}\right)^2 \hat{\sigma}_h^{\ 2}}$$

$$= \pm\sqrt{(1026.254)^2(0.002)^2 + (205.251)^2(0.018)^2} = \pm\ 4.226 \text{ cm}^3$$

$\Rightarrow V = 7183.775 \pm 4.226 \text{ cm}^3$

**EXAMPLE 2.6:**

Two sides and the included angle of a triangular land parcel were measured with the following results: a = 45.12 ± 0.05 m, b = 38.64 ± 0.03 m, and θ = 52° 15′ ± 30″. Calculate the area of the land parcel and its standard error.

**SOLUTION:**

The area of the triangle is given by the following relationship:

$$A = \frac{1}{2} ab \sin\theta$$
$$= \frac{1}{2} \text{ x } 45.12 \text{ x } 38.64 \sin(52^\circ\ 15')$$
$$= 689.26\ m^2$$

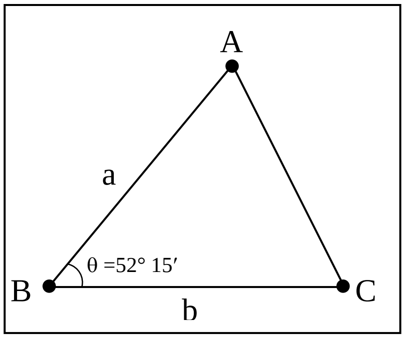


**FIGURE 2.7:** A triangular land parcel.

The standard error of the area is (from Equation 2.12):

$$\hat{\sigma}_A = \pm\sqrt{\left(\frac{\partial A}{\partial a}\right)^2 \hat{\sigma}_a^{\ 2} + \left(\frac{\partial A}{\partial b}\right)^2 \hat{\sigma}_b^{\ 2} + \left(\frac{\partial A}{\partial \theta}\right)^2 \hat{\sigma}_\theta^{\ 2}}$$

$$\hat{\sigma}_a = \pm\ 0.05 \text{ m}, \quad \hat{\sigma}_b = \pm\ 0.03 \text{ m}, \quad \hat{\sigma}_\theta = \pm\ \frac{30}{3600} \text{ x } \frac{\pi}{180} = 1.454\text{x}10^{-4} \text{ radian}$$

$$\frac{\partial A}{\partial a} = \frac{1}{2} b \sin\theta = \frac{1}{2} \text{ x } 38.64 \sin(52^\circ\ 15') = 15.28 \text{ m}$$

$$\frac{\partial A}{\partial b} = \frac{1}{2} a \sin\theta = \frac{1}{2} \text{ x } 45.12 \sin(52^\circ\ 15') = 17.84 \text{ m}$$

$$\frac{\partial A}{\partial \theta} = \frac{1}{2} ab \cos\theta = \frac{1}{2} \text{ x } 45.12 \text{ x } 38.64 \cos(52^\circ\ 15') = 533.68\ m^2$$

$$\Rightarrow \hat{\sigma}_A = \pm\sqrt{(15.28)^2(0.05)^2 + (17.84)^2(0.03)^2 + (533.68)^2(1.454\text{x}10^{-4})^2}$$
$$= \pm\ 0.94\ m^2$$

## 2.9 WEIGHTS AND WEIGHTED MEAN

Sometimes, one measurement (observation) of a series may be more reliable than another. Such an observation should exert greater influence upon the calculation of the results. The degree of reliability is commonly termed the *weight* of the measurement. This is merely the relative value of that observation to the others of the series.

When calculating the mean value of some quantity from two or more sets of observations, it is logical to give consideration to the calculated precision of each of the sets. The weights are taken to be inversely proportional to the square of the standard error, that is.

$$\frac{w_1}{w_2} = \frac{\sigma_2^{\,2}}{\sigma_1^{\,2}}$$

Or $$w_i \propto \frac{1}{\sigma_i^{\,2}} \Rightarrow w_i = \frac{k}{\sigma_i^{\,2}}$$

Let $k = \sigma_0^{\,2}$

$$\Rightarrow \quad w_i = \frac{\sigma_0^{\,2}}{\sigma_i^{\,2}} \qquad \ldots\ldots\ldots (2.13)$$

$\sigma_0$ is called the standard error of unit weight because if the standard error $\sigma_i$ of a measurement is equal to $\sigma_0$, then it has a weight of 1.

Since the weight is inversely proportional to the square of the standard error $\sigma_i$ , then, the more precise the measurement is, the smaller will be its standard error and the larger will be its weight.

Now, let $x_1, x_2, x_3 \ldots x_n$ be *n* independent measurements of a quantity, and let $\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3 \ldots \hat{\sigma}_n$ be the corresponding standard errors of these measurements. This means that the measurements are assumed to be made with different precision. It can be shown that the most probable or accepted value $(\hat{x})$ of the quantity is given by the weighted mean of these *n* measurements; that is:

$$\hat{x} = \frac{w_1x_1 + w_2x_2 + \ldots w_nx_n}{w_1 + w_2 + \ldots w_n} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} \qquad \ldots\ldots\ldots (2.14)$$

Moreover, an estimate of the standard error of the weighted mean $(\hat{\sigma}_{\hat{x}})$ can be computed as follows:

$$\hat{\sigma}_{\hat{x}} = \pm \frac{\sigma_0}{\sqrt{\sum_{i=1}^{n} w_i}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2.15)$$

| | |
|---|---|
| **Problem:** | Use the law of propagation of random errors to prove that the previous equation is correct. |

**EXAMPLE 2.7:**

Compute the weighted mean ($\hat{\ell}$) and the estimated standard error of the weighted mean ($\sigma_{\hat{\ell}}$) for the following four independent measurements of a distance:

$\ell_1$ = 2746.34 ±0.02 ft $\qquad$ $\ell_2$ = 2746.38 ± 0.06 ft

$\ell_3$ = 2746.26 ± 0.05 ft $\qquad$ $\ell_4$ = 2746.31 ± 0.04 ft

**SOLUTION:**

Let $\sigma_0$ = ± 0.06 ft, then:

$$w_1 = \left(\frac{0.06}{0.02}\right)^2 = 9, \qquad w_3 = \left(\frac{0.06}{0.05}\right)^2 = 1.44$$

$$w_2 = \left(\frac{0.06}{0.06}\right)^2 = 1, \qquad w_4 = \left(\frac{0.06}{0.04}\right)^2 = 2.25$$

$$\hat{\ell} = \frac{2746.34 \text{ x } 9 + 2746.38 \text{ x } 1 + 2746.26 \text{ x } 1.44 + 2746.31 \text{ x } 2.25}{9 + 1 + 1.44 + 2.25}$$

= 2746.33 ft

$$\sigma_{\hat{\ell}} = \pm \frac{0.06}{\sqrt{13.69}} = \pm 0.02 \text{ ft}$$

| | |
|---|---|
| **Problem:** | Prove that the weighted mean and its standard error will not change regardless of the chosen value for $\sigma_0$. Support your proof by choosing a different value for $\sigma_0$ in the previous example and solve it again. |

## 2.10 SIGNIFICANT FIGURES

The significant figures in a number are those digits with known values. They are identified by proceeding from left to right, beginning with the first non-zero digit and ending with the last digit of the number. The following rules may be helpful:

1 - All non-zero digits are significant
2 - Zeros at the beginning of a number merely indicate the position of the decimal point. They are not significant.
3 - Zeros between digits are significant
4 - Zeros at the end of a decimal number are significant.

**<u>Examples</u>:**
a - 456.300 has six significant figures
b - 0.0036 has two significant figures
c - 6.000350 has seven significant figures
d - 54.0 has three significant figures

The subject of significant figures is important in both fieldwork and office computations. Since neither the measurements nor the quantities mathematically deduced from them could be exact, it is essential to use the appropriate number of significant figures to express a final meaningful result.

When there are more significant figures in a quantity than are required, the number is rounded off to the number of places needed. The following points should be taken into consideration when deciding upon the number of significant figures in surveying operations:

1) Any calculated value should correspond with its standard error. For example, if the standard error of a distance is ±0.002 m, the value of the distance should be reported to the third decimal place.

2) The number of decimal places in a measurement should not exceed the accuracy of the fieldwork. For example, if a distance can be measured with a tape which can read up to 0.001 m, then it is not reasonable to report a distance with more than three decimal places.

3) When performing addition, subtraction, multiplication or division, the answer can not be more precise than the least precise number included in the mathematical operation.

For example:

$$\begin{array}{r} 24.217 \\ +\ 468.46 \\ +1563.1 \\ \hline 2055.777 \end{array}$$

The sum must be rounded off to 2055.8 because 1563.1 has only one decimal place.

## PROBLEMS

**2.1** For the following set of repeated measurements of a distance: 576.39, 576.29, 576.31, 576.34, 576.35, 576.30, 576.33, 576.27, 576.34 and 576.30 m.

a. Compute the mean, standard deviation and estimated standard error of the mean.
b. Check for the presence of any blunders. Reject blunders if any and repeat calculations in part (a).
c. Choose an appropriate interval and plot a histogram for the errors. (for simplicity use 5 intervals)

**2.2** A distance is measured to be 456.31 m with an estimated standard error of $\pm$ 0.05 m. Compute for this measured distance:

a. The probable error. b. The maximum error.
c. The relative precision at $1\sigma$. d. The relative precision at $3\sigma$.

**2.3** The length and width of a rectangular field are 5420 ft and 1510 ft respectively. If the area of the field must be determined with a standard error of $\pm$ 0.1 acre, determine the relative precision at $1\sigma$ with which the length and width of the field must be measured.

**2.4** A tract of land is trapezoidal in shape with the following dimensions:

$\ell_1 = 472.3 \pm 0.1$ m
$\ell_2 = 583.7 \pm 0.3$ m
$h = 241.8 \pm 0.2$ m

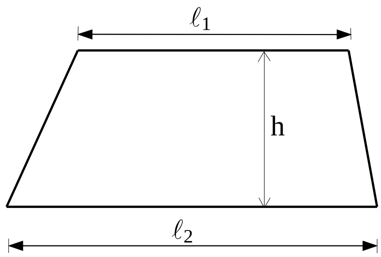


**FIGURE 2.8:** A trapezoidal land parcel

Compute the area of the tract and its estimated standard error.

**2.5** A line was carefully measured 10 times on 3 different days. The mean and the estimated standard error of each day's measurement were computed to be as follows:

| Day | Mean | Estimated Standard Error |
|---|---|---|
| 1 | 2815.46 m | ± 0.05 m |
| 2 | 2816.72 m | ±0.03 m |
| 3 | 2816.38 m | ±0.02 m |

Compute:

a. The weighted mean of the three measurements.

b. The estimated standard error of the weighted mean.

**2.6** Given below are the elevations and the RMS errors measured from two surveys for two subsidence-monitoring points:

| POINT | ELEVATIONS | |
|---|---|---|
| # | JUNE 1974 | JUNE 1984 |
| 101 | 563.14 ± 0.03 m | 563.01 ± 0.06 m |
| 102 | 579.26 ± 0.04 m | 579.05 ± 0.05 m |

a. Compute for each point the change in elevation, the RMS error of the change, and the maximum expected survey error in the change. Make a table.

b. Which point has an elevation change exceeding the maximum expected survey error?

**2.7** Derive equation (2.8).