# LINEAR PROGRAMMING APPROACHES TO AGGREGATE PLANNING

Linear programming is suitable to determine the best aggregate plan.

Recall that linear programming assumes all variables are continuously divisible. Thus the solution may be to produce 2142.3 units next month. Aggregate units are fictitious, so it probably does not matter. Also, rounding large numbers will likely be acceptable. However, rounding 1.5 workers could be problematic.

## PARAMETERS AND VARIABLES

$T$ = planning-horizon length, in periods
$t$ = index of periods, $t = 1, 2, \ldots, T$
$D_t$ = forecasted number of units demanded in period $t$
$n_t$ = number of units that can be made by one worker in period $t$
$C_t^P$ = cost to produce one unit in period $t$
$C_t^W$ = cost of one worker in period $t$
$C_t^H$ = cost to hire one worker in period $t$
$C_t^L$ = cost to lay off one worker in period $t$
$C_t^I$ = cost to hold one unit in inventory for period $t$
$C_t^B$ = cost to backorder one unit for period $t$

$P_t$ = number of units produced in period $t$
$W_t$ = number of workers available in period $t$
$H_t$ = number of workers hired in period $t$
$L_t$ = number of workers laid off in period $t$
$I_t$ = number of units held in inventory at the end of period $t$
$B_t$ = number of units backordered at the end of period $t$

## CONSTRAINTS

Constraints on capacity, work force, and material.

- The size of the work force limits the number of units we can produce.
- The number of workers available is a function of the number we start with and how many we hire and lay off.
- Net inventory this period = net inventory last period + productions this period - demand this period.

## COSTS

The cost for any plan is the sum of productions costs, hiring and lay-off costs, inventory holding costs, and backorder costs over all periods.

## A MODEL

$$\text{Minimize} \sum_{t=1}^{T} (C_t^P P_t + C_t^W W_t + C_t^H H_t + C_t^L L_t + C_t^I I_t + C_t^B B_t)$$

subject to

$$P_t \leq n_t W_t \qquad t = 1, 2, \ldots, T$$

$$W_t = W_{t-1} + H_t - L_t \qquad t = 1, 2, \ldots, T$$

$$I_t - B_t = I_{t-1} - B_{t-1} + P_t - D_t \qquad t = 1, 2, \ldots, T$$

$$P_t, W_t, H_t, L_t, I_t \geq 0 \qquad t = 1, 2, \ldots, T$$

**Example**: *Precision Gears Inc*. produces 41,383 gears/year. There are 260 working days and 40 workers ⇒ ≈ 4 gears/worker-day.

Production costs, excluding labor, do not change over the planning horizon and thus are ignored. A unit produced but not sold in a month is counted as inventory for that entire month (end-of-month inventory). End of the month inventory holding cost is $5 per gear per month. At the beginning of each month, new workers can be hired at a cost of $450 per worker. Existing workers can be laid off at a cost of $600 per worker. Wages and benefits for a worker are $15 per hour, all workers are paid for eight hours per day, and there are currently 35 workers at Precision Inc.

**Aggregate demand forecast for gears**

| **Month** | Jan | Feb | March | April | May | June | Total |
|---|---|---|---|---|---|---|---|
| Demand | 2760 | 3320 | 3970 | 3540 | 3180 | 2900 | 19,670 |

Minimize
$$\begin{aligned}&2520W_1 + 2400W_2 + 2760W_3 + 2520W_4 + 2640W_5 + 2640W_6\\ &+ 450H_1 + 450H_2 + 450H_3 + 450H_4 + 450H_5 + 450H_6\\ &+ 600L_1 + 600L_2 + 600L_3 + 600L_4 + 600L_5 + 600L_6\\ &+ 5I_1 + 5I_2 + 5I_3 + 5I_4 + 5I_5 + 5I_6\end{aligned}$$

Subject to

Production-capacity constraints
$$\begin{aligned}P_1 &\le 84W_1\\ P_2 &\le 80W_2\\ P_3 &\le 92W_3\\ P_4 &\le 84W_4\\ P_5 &\le 88W_5\\ P_6 &\le 88W_6\end{aligned}$$

Work-force constraints
$$\begin{aligned}W_1 &= 35 + H_1 - L_1\\ W_2 &= W_1 + H_2 - L_2\\ W_3 &= W_2 + H_3 - L_3\\ W_4 &= W_3 + H_4 - L_4\\ W_5 &= W_4 + H_5 - L_5\\ W_6 &= W_5 + H_6 - L_6\end{aligned}$$

Inventory-balance constraints
$$\begin{aligned}I_1 &= \quad + P_1 - 2760\\ I_2 &= I_1 + P_2 - 3320\\ I_3 &= I_2 + P_3 - 3970\\ I_4 &= I_3 + P_4 - 3540\\ I_5 &= I_4 + P_5 - 3180\\ I_6 &= I_5 + P_6 - 2900\end{aligned}$$

$$\begin{aligned}&P_1, P_2, P_3, P_4, P_5, P_6, W_1, W_2, W_3, W_4, W_5, W_6,\\ &H_1, H_2, H_3, H_4, H_5, H_6, L_1, L_2, L_3, L_4, L_5, L_6,\\ &I_1, I_2, I_3, I_4, I_5, I_6 \ge 0.\end{aligned}$$

**Linear programming solution**

| Month | Production | Inventory | Hired | Laid off | Workers |
|---|---|---|---|---|---|
| January | 2940.00 | 180.0 | 0.00 | 0.00 | 35.00 |
| February | 3232.86 | 92.86 | 5.41 | 0.00 | 40.41 |
| March | 3877.14 | 0.00 | 1.73 | 0.00 | 42.14 |
| April | 3540.00 | 0.00 | 0.00 | 0.00 | 42.14 |
| May | 3180.00 | 0.00 | 0.00 | 6.00 | 36.14 |
| June | 2900.00 | 0.00 | 0.00 | 3.18 | 32.95 |

Total cost = $600,191.60

We assumed that only full-time workers were available, we must adjust the solution.

**Rounding the linear programming solution**

| | | January | February | March | April | May | June | Total |
|---|---|---|---|---|---|---|---|---|
| 1 | Units/worker | 84 | 80 | 92 | 84 | 88 | 88 | 516 |
| 2 | Workers | 35 | 41 | 42 | 42 | 36 | 33 | 229 |
| 3 | Capacity | 2940 | 3280 | 3864 | 3528 | 3168 | 2904 | 19,684 |
| 4 | Demand | 2760 | 3320 | 3970 | 3540 | 3180 | 2900 | 19,670 |
| 5 | Capacity − demand | 180 | −40 | −106 | −12 | −12 | 4 | 14 |
| 6 | Cumulative difference | 180 | 140 | 34 | 22 | 10 | 14 | 14 |
| 7 | Produced | 2930 | 3280 | 3864 | 3528 | 3168 | 2900 | 19,670 |
| 8 | Net inventory | 170 | 130 | 24 | 12 | 0 | 0 | 336 |

The rounded plan has a total cost of $600,750 compared to the linear programming solution.

Some advantages of linear programming are:

- dual variables (or shadow-prices).
- reduced cost information
- Impose explicit restrictions, or scenarios ($B_T$=0).

See the LINDO solution.

## **Some extensions to the LP Model:**

Inventory space limitations:

$$I_t \leq I_t^u$$

Safety stock considerations:

$$I_t^l \leq I_t \leq I_t^u$$

Layoff restrictions:

$$L_t \leq 0.05\ W_t$$

Training time considerations:

$$W_t = W_{t-1} + H_{t-1} - L_t$$

# ADVANCED PRODUCTION PLANNING MODELS

## A MULTIPLE PRODUCT AGGREGATE PLANNING MODEL

$T$ = horizon length, in periods
$N$ = number of products
$t$ = index of periods, $t = 1, 2, \ldots, T$
$i$ = index of products, $i = 1, 2, \ldots, N$
$D_{it}$ = forecasted number of units demanded for product $i$ in period $t$
$n_{it}$ = number of units of product $i$ that can be made by one worker in period $t$
$C_{it}^{P}$ = cost to produce one unit of product $i$ in period $t$
$C_t^W$ = cost of one worker in period $t$
$C_t^H$ = cost to hire one worker in period $t$
$C_t^L$ = cost to lay off one worker in period $t$
$C_{it}^I$ = cost to hold one unit of product $i$ in inventory for period $t$

The decision variables are

$P_{it}$ = number of units of product $i$ produced in period $t$
$W_t$ = number of workers available in period $t$
$H_t$ = number of workers hired in period $t$
$L_t$ = number of workers laid off in period $t$
$I_{it}$ = number of units of product $i$ held in inventory at the end of period $t$

The linear programming formulation is

$$\text{Minimize} \sum_{t=1}^{T} \sum_{i=1}^{N} (C_{it}^P P_{it} + C_t^W W_t + C_t^H H_t + C_t^L L_t + C_{it}^I I_{it})$$

$$\text{subject to} \qquad \sum_{i=1}^{N} \left(\frac{1}{n_{it}}\right) P_{it} \le W_t \qquad t = 1, 2, \ldots, T$$

$$W_t = W_{t-1} + H_t - L_t \qquad t = 1, 2, \ldots, T$$

$$I_{it} = I_{it-1} + P_{it} - D_{it} \qquad t = 1, 2, \ldots, T; \quad i = 1, 2, \ldots, N$$

$$P_{it}, W_t, H_t, L_t, I_{it} \ge 0 \qquad t = 1, 2, \ldots, T; \quad i = 1, 2, \ldots, N$$

This model is similar to an aggregate planning model. However, we now have production and inventory variables for each product as well as for each period. Also, there are material balance constraints for every product and every period.

**Example 5-2. Carolina Hardwood Product Mix.** Carolina Hardwood produces three types of dining tables. There are currently 50 workers; new workers can be hired, and existing workers can be laid off. During the next four quarters, the cost of hiring one worker is 420, 410, 420, and 405, respectively. The cost to lay off one worker is 800, 790, 790, and 800. The cost of one worker per quarter is 600, 620, 620, and 610. The initial inventory is 100 units for table 1, 120 units for table 2, and 80 units for table 3. The number of units that can be made by one worker per quarter is 200, 220, 210, and 200 for table 1. They are 300, 310, 300, and 290 for table 2 and 260, 255, 250, and 265 for table 3. Forecasted demand, unit cost, and holding cost per unit are:

| | Demand | | | Unit cost | | | Holding cost | | |
|---|---|---|---|---|---|---|---|---|---|
| Quarter | Table 1 | Table 2 | Table 3 | Table 1 | Table 2 | Table 3 | Table 1 | Table 2 | Table 3 |
| 1 | 3500 | 5400 | 4500 | 120 | 150 | 200 | 10 | 12 | 12 |
| 2 | 3100 | 5000 | 4200 | 125 | 150 | 210 | 9 | 11 | 12 |
| 3 | 3000 | 5100 | 4100 | 120 | 145 | 205 | 10 | 12 | 11 |
| 4 | 3400 | 5500 | 4600 | 125 | 148 | 205 | 10 | 11 | 11 |

Minimize $600W_1 + 620W_2 + 620W_3 + 610W_4 + 420H_1 + 410H_2 + 420H_3$
$+ 405H_4 + 800L_1 + 790L_2 + 790L_3 + 800L_4 + 120P_{11} + 150P_{21}$
$+ 200P_{31} + 125P_{12} + 150P_{22} + 210P_{32} + 120P_{13} + 145P_{23} + 205P_{33}$
$+ 125P_{14} + 148P_{24} + 205P_{34} + 10I_{11} + 12I_{21} + 12I_{31} + 9I_{12}$
$+ 11I_{22} + 12I_{32} + 10I_{13} + 12I_{23} + 11I_{33} + 10I_{14} + 11I_{24} + 11I_{34}$

subject to

$$P_{11}/200 + P_{21}/300 + P_{31}/260 \le W_1$$
$$P_{12}/220 + P_{22}/310 + P_{32}/255 \le W_2$$
$$P_{13}/210 + P_{23}/300 + P_{33}/250 \le W_3$$
$$P_{14}/200 + P_{24}/290 + P_{34}/265 \le W_4$$

$$W_1 = 50 + H_1 - L_1$$
$$W_2 = W_1 + H_2 - L_2$$
$$W_3 = W_2 + H_3 - L_3$$
$$W_4 = W_3 + H_4 - L_4$$

$$I_{11} = 100 + P_{11} - 3500$$
$$I_{21} = 120 + P_{21} - 5400$$
$$I_{31} = 80 + P_{31} - 4500$$
$$I_{12} = I_{11} + P_{12} - 3100$$
$$I_{22} = I_{21} + P_{22} - 5000$$
$$I_{32} = I_{31} + P_{32} - 4200$$
$$I_{13} = I_{12} + P_{13} - 3000$$
$$I_{23} = I_{22} + P_{23} - 5100$$
$$I_{33} = I_{32} + P_{33} - 4100$$
$$I_{14} = I_{13} + P_{14} - 3400$$
$$I_{24} = I_{23} + P_{24} - 5500$$
$$I_{34} = I_{33} + P_{34} - 4600$$

$$P_{it}, I_{it}, W_t, H_t, L_t \ge 0 \qquad t = 1, \ldots, 4; \quad i = 1, \ldots, 3$$

**Optimum solution is:**

| Quarter | Production | | | Inventory | | |
|---|---|---|---|---|---|---|
| | Table 1 | Table 2 | Table 3 | Table 1 | Table 2 | Table 3 |
| 1 | 3400 | 5280 | 4420 | 0 | 0 | 0 |
| 2 | 3100 | 5000 | 4200 | 0 | 0 | 0 |
| 3 | 3000 | 5100 | 4100 | 0 | 0 | 0 |
| 4 | 3400 | 5500 | 4600 | 0 | 0 | 0 |

| Quarter | Workforce | Hired | Laid off |
|---|---|---|---|
| 1 | 51.60 | 1.60 | 0.00 |
| 2 | 47.69 | 0.00 | 3.91 |
| 3 | 47.69 | 0.00 | 0.00 |
| 4 | 53.32 | 5.64 | 0.00 |

*Objective function value* = $8,354,166

## SUMMARY

Aggregate planning focuses on intermediate-range production planning problems. At this level of planning, usually an aggregate unit rather than an individual product is considered. The aggregate unit is defined by some measure common to all products, such as production costs, capacity change costs and inventory costs.

Two major approaches are used to generate an aggregate production plan: spreadsheet methods and quantitative methods.

Spreadsheet methods are trial and error approaches.  Typical strategies are zero inventory, level production (constant work force), and mixed strategies.  The result is a feasible aggregate plan that usually projects a realistic cost.

The quantitative methods used in aggregate planning are variations of linear programming.  Under given assumptions, these methods yield an optimal aggregate plan.  The linear programming models can be extended to more general planning situations, including multiple products and multiple processes.

Aggregate planning generates a production plan in aggregate units from forecasts and the aggregate plan is disaggregated into a plan for individual items.  This plan becomes the master production schedule (MPS).