CHAPTER 2

ERRORS IN SURVEYING

2.1 INTRODUCTION

Let us measure a distance under the following circumstances:

- Perform the measurement on a beautiful day (no wind, comfortable temperature and so on) and write down the result.
- Repeat the measurement immediately on the same day with the same tape.
- Repeat the measurement on a windy day.
- Repeat the measurement on a hot day.
- Repeat the measurement on a cold day.
- Repeat the measurement under the previous conditions but with a different tape.
- Repeat the measurement under the previous conditions, but two different people are asked this time to do the job.

Three sources for errors:

- The imperfections of the instruments,
- The fallibility of the human operator, and
- The uncontrollable nature of the environment.

- The true value of a measurable variable is never known.

2.2 ERRORS IN SURVEYING MEASUREMENTS

- The *true error* in a surveying measurement is defined as the difference between the measured value of a parameter and its true value.
- Let e_i = true error

  x_i = measured value

  x = true value

  $\Rightarrow e_i = x_i - x$
- Since the true value (x) is never known, e_i can never be calculated.
- Therefore we compare with another acceptable approximate of the true value.

Let $\hat{x}$ represent such a value. Then, an estimate (v_i) of the true error (e_i) is:

$$v_i = x_i - \hat{x} \quad \text{(called residual error)}$$

- Error and Correction:

  Correction = - Error

- In general, errors in surveying measurements can be divided into three different types:

1. Blunders (mistakes),
2. Systematic errors, and
3. Random errors (compensating or accidental errors).

2.2.1 BLUNDERS (MISTAKES)

- mistakes caused by human carelessness, fatigue and haste.

- Blunders can be positive or negative, large or small and their occurrence is unpredictable.

- An example of blunders:
  Recording 18 instead of 81.

- Blunders need to be eliminated by careful work and by using mathematical techniques as will be explained later in this course.

2.2.2 SYSTEMATIC ERRORS

- Mostly caused by the maladjustment of the surveying instruments and by the uncontrollable nature of the environment.
- behave according to a particular system or physical law of nature, which may or may not be known. When the law of occurrence is known, systematic errors can be calculated and eliminated from the measurements.
- Examples: temperature and tape length error.

2.2.3 RANDOM ERRORS (COMPENSATING OR ACCIDENTAL ERRORS)

- caused by imperfections of the measuring instruments, imperfections of the surveyor to make an exact measurement, and the uncontrollable variations in the environment.
- Random errors have the following characteristics:

1. Positive and negative errors of the same magnitude occur with the same frequency.
2. Small errors occur more frequently than large ones.
3. Very large errors seldom occur.
4. The mean of an infinite number of observations is the true value.

Distance Measurements	Residuals
d_1	$v_1 = d_1 - \bar{d}$
d_2	$v_2 = d_2 - \bar{d}$
d_3	$v_3 = d_3 - \bar{d}$
d_4	$v_4 = d_4 - \bar{d}$
.	.
.	.
.	.
.	.
d_n	$v_n = d_n - \bar{d}$

$$\bar{d} = \frac{\sum d_i}{n}$$

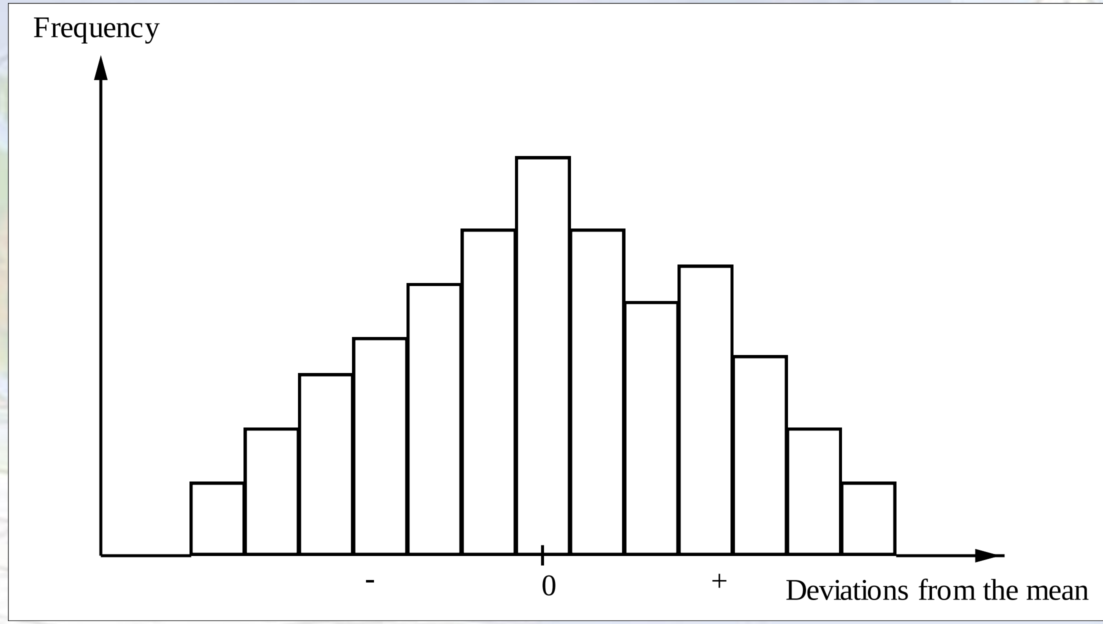


FIGURE 2.1: A histogram which shows the distribution of random errors

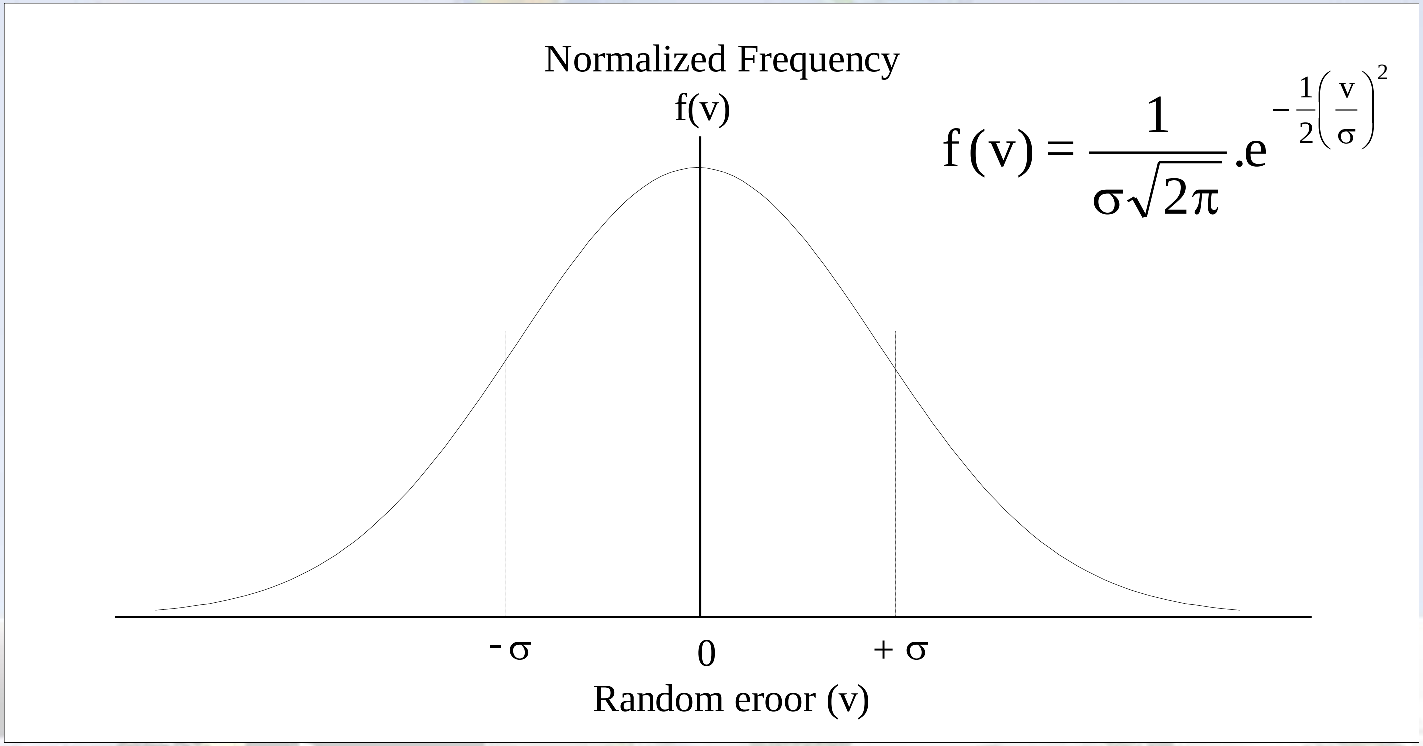


FIGURE 2.2: Normal curve of error.

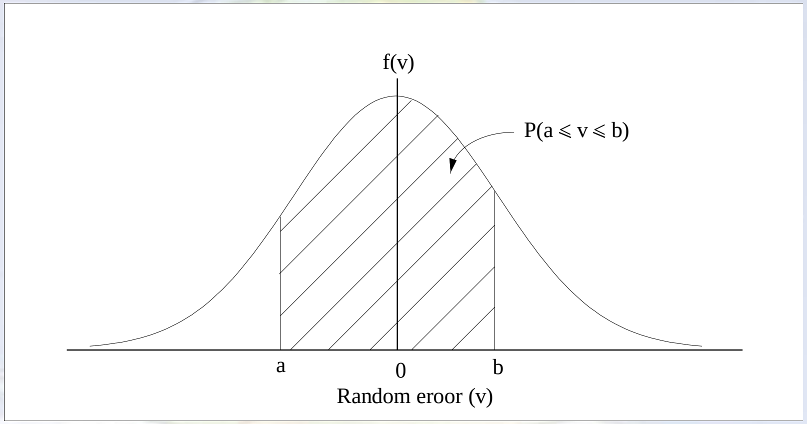
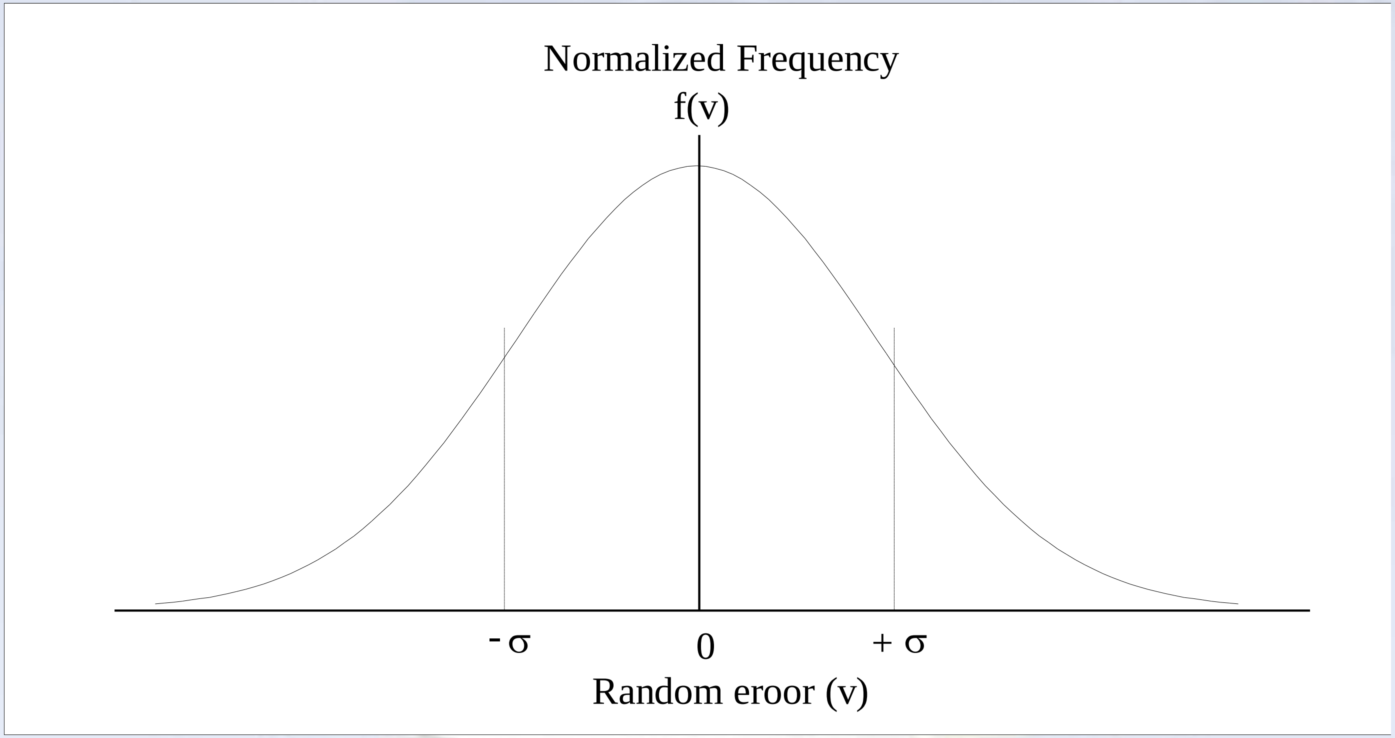


FIGURE 2.3: Probability of random errors.

$$P(a \le v \le b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} . e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^2} . dv$$

Error Range	Probability (%)
$\pm$ 0.6745 σ	50.0
$\pm$ 1.00 σ	68.3
$\pm$ 1.6449 σ	90.0
$\pm$ 2.00 σ	95.4
$\pm$ 3.00 σ	99.7



For a distance D, given $\sigma = \pm 2$ cm, compute $P(-4 \text{ cm} < v < 6 \text{ cm})$.

Solution:

$P(-4 \text{ cm} < v < 6 \text{ cm}) = 95.4\%/2 + 99.7\%/2 = 97.55\%$

2.3 MEAN, STANDARD DEVIATION AND STANDARD ERROR OF THE MEAN

Let $x_1, x_2, x_3 \ldots x_n$ be n repeated measurements of the same quantity that are measured with the same degree of care. Then:

1) The simple mean $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$

2) An estimate of the standard error $\hat{\sigma}_x$ of *one measurement* of the quantity is:

$$\hat{\sigma}_x = \pm\sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

3) An estimate of the standard error of the *mean of the n measurements* ($\hat{\sigma}_{\bar{x}}$) is:

$$\hat{\sigma}_{\bar{x}} = \pm\sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n(n-1)}}$$

Or

$$\hat{\sigma}_{\bar{x}} = \frac{\hat{\sigma}_x}{\sqrt{n}}$$

$$
\begin{array}{cccc}
X_1 & X_1 & X_1 & \\
X_2 & X_2 & X_2 & \\
X_3 & X_3 & X_3 & \cdots\cdots \\
\vdots & \vdots & \vdots & \\
X_n & X_n & X_n & \\
\hline
\bar{X}_1 = \frac{\sum X_i}{n} & \bar{X}_2 & \bar{X}_3 & \bar{X} = \frac{\sum \bar{X}_i}{m}
\end{array}
$$

$$\hat{\sigma}_{\bar{X}} = \pm\sqrt{\frac{\sum(\bar{X}_i - \bar{X})^2}{m-1}} = \frac{\hat{\sigma}_{X_i}}{\sqrt{n}}$$

$$\bar{X} = \frac{\sum X_i}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$\bar{X} = \frac{X_1}{n} + \frac{X_2}{n} + \cdots + \frac{X_n}{n}$$

Apply the Law of propagation of random errors:

$$\hat{\sigma}_{\bar{X}}^2 = \frac{1}{n^2}\hat{\sigma}_{X_1}^2 + \frac{1}{n^2}\hat{\sigma}_{X_2}^2 + \cdots + \frac{1}{n^2}\hat{\sigma}_{X_n}^2$$

$$= \frac{1}{n^2}\left(\hat{\sigma}_{X_1}^2 + \hat{\sigma}_{X_2}^2 + \cdots + \hat{\sigma}_{X_n}^2\right)$$, but $\hat{\sigma}_{X_1} = \hat{\sigma}_{X_2} = \cdots = \hat{\sigma}_{X_n}$ (Same degree of care)

$$\Rightarrow \hat{\sigma}_{\bar{X}}^2 = \frac{1}{n^2}\left(n \cdot \hat{\sigma}_{X_i}^2\right) = \frac{\hat{\sigma}_{X_i}^2}{n}$$

$$\Rightarrow \boxed{\hat{\sigma}_{\bar{X}} = \frac{\hat{\sigma}_{X_i}}{\sqrt{n}}}$$

2.4 PROBABLE AND MAXIMUM ERRORS

- The *probable error* of a measurement is defined to be equal to 0.6745σ (50% probability).

- The *maximum error* in a measurement is defined as being equal to 3σ (99.7% probability).

- **Example**:

  If the standard error of an angle measurement is ± 3.0 seconds, then,

  The probable error = ± (0.6745 x 3.0) = ± 2.0 seconds

  The maximum error = ± (3 x 3.0) = ± 9.0 seconds

Example on how the maximum error is used to detect blunders:

Measurement (m)	First Iteration $v_i = d_i - \bar{d}$ (m)	Second Iteration $v_i = d_i - \bar{d}$ (m)
58.78	0.03	-0.03
58.83	0.08	0.02
58.80	0.05	-0.01
58.85	0.10	0.04
58.18	-0.57	blunder $\Rightarrow$ rejected
58.77	0.02	-0.04
58.79	0.04	-0.02
58.80	0.05	-0.01
58.81	0.06	0.00
58.82	0.07	0.01
58.79	0.04	-0.02
58.82	0.07	0.01

First Iteration: (n = 12)

- Mean = 58.75 m
- Standard deviation = ± 0.18 m
- Maximum error of a single measurement = ± 3 x 0.18 = ± 0.54 m

$\Rightarrow$ Reject measurement 58.18 m (possibly the surveyor recorded 58.18 m instead of 58.81 m)

Second Iteration: (n = 11)

- Mean = 58.81 m
- Standard deviation = ± 0.02 m
- Maximum error of a single measurement = ± 3 x 0.02 = ± 0.06 m

$\Rightarrow$ No more measurements are rejected.

2.5 PRECISION AND ACCURACY

Precision: A measurement is considered to have high precision if it has a small standard deviation.

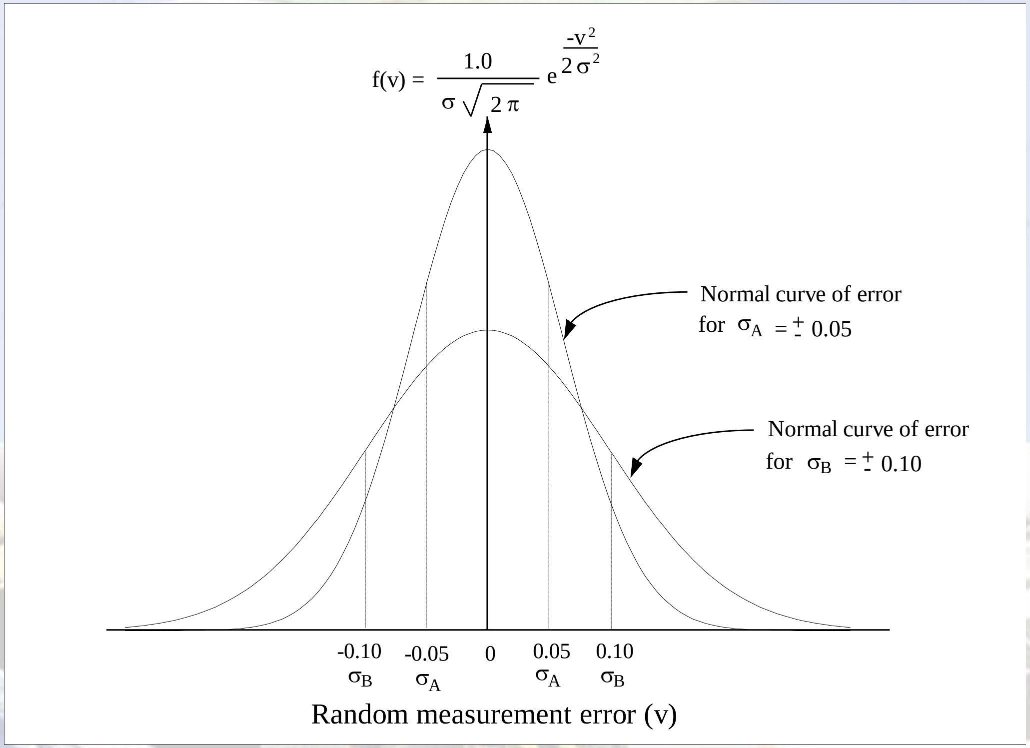


FIGURE 2.4: Standard error and the distribution of random errors.

Accuracy: A measurement is considered to have high accuracy if it is close to the true value.

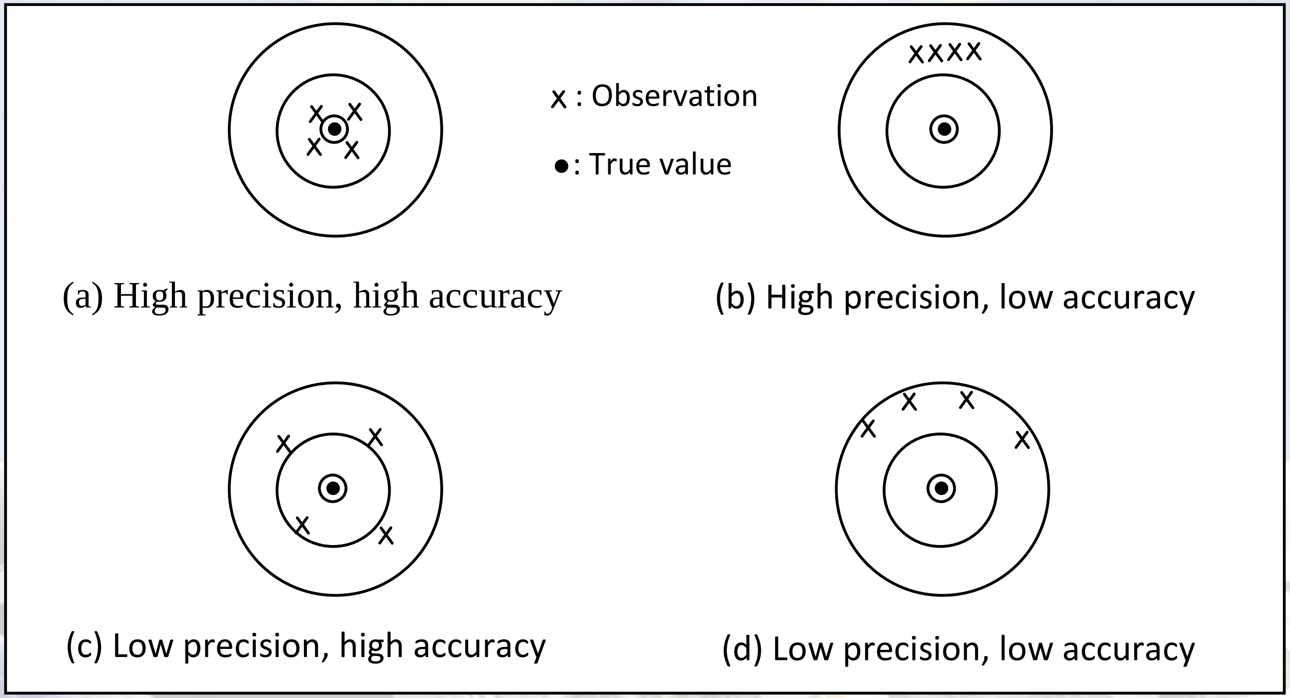


FIGURE 2.5: Possible combinations of precision and accuracy.

EXAMPLE:

A distance was measured by two independent parties, with the following results:

Party A: D_A = 40.64 $\pm$ 0.04 m

Party B: D_B = 40.56 $\pm$ 0.02 m

If by some means the true value was known to be 40.61 m, Compare between the two teams in terms of precision and accuracy.

SOLUTION:

$\hat{\sigma}_B = \pm 0.02$ m $< \hat{\sigma}_A = \pm 0.04$ m

$\Rightarrow$ Party B measurement is more precise than Party A.

Absolute error in party A measurement = $|40.64 - 40.61| = 0.03\ \text{m}$

Absolute error in party B measurement = $|40.56 - 40.61| = 0.05\ \text{m}$

$0.05 > 0.03$ $\Rightarrow$ Party B measurement is less accurate than Party A.

Result: Party B measurement is more precise but less accurate than Party A measurement.

2.6 RELATIVE PRECISION

- Relative precision is a term that is commonly used to describe the precision of *distance measurement* in surveying.

- Suppose that a distance D is measured with a standard error σ_D, then:

Relative precision of the measured distance at 1 σ = $\frac{1}{D/\sigma_D}$

- The denominator is usually approximated to one or two non-zero digits.

Example:

For D = 60.64 $\pm$ 0.04 m,

Relative precision at 1 σ level = $\frac{1}{\frac{60.64}{0.04}} = \frac{1}{1516} \cong \frac{1}{1500}$

2.7 REPEATED MEASUREMENTS

$$\hat{\sigma}_{\bar{x}} = \frac{\hat{\sigma}_x}{\sqrt{n}}$$

$$\Rightarrow \qquad n = \left(\frac{\hat{\sigma}_x}{\hat{\sigma}_{\bar{x}}}\right)^2$$

For example: Suppose that an angle can be measured with $\hat{\sigma}_x = \pm 3''$ in one repetition by using a certain instrument, then the number of repetitions required to determine the angle with $\hat{\sigma}_{\bar{x}} = \pm 1''$ is:

$$n = \left(\frac{3}{1}\right)^2 = 9$$

2.8 PROPAGATION OF RANDOM ERRORS

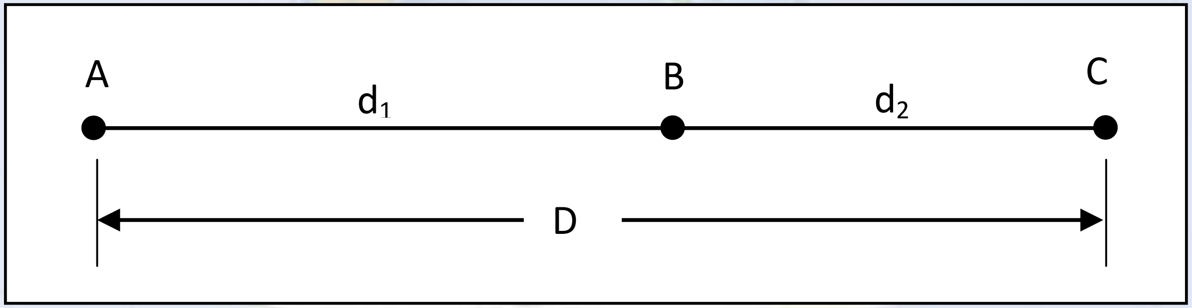
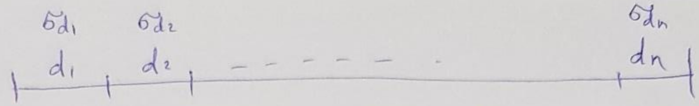


FIGURE 2.6: Measuring a long distance in two sections.

$$y = F(x_1, x_2, \ldots x_n)$$

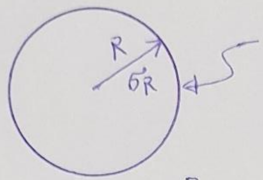
$$\hat{\sigma}_y^2 = \left(\frac{\partial F}{\partial x_1}\right)^2 \hat{\sigma}_{x_1}^2 + \left(\frac{\partial F}{\partial x_2}\right)^2 \hat{\sigma}_{x_2}^2 + \ldots + \left(\frac{\partial F}{\partial x_n}\right)^2 \hat{\sigma}_{x_n}^2$$

①



$$D = d_1 + d_2 + \dots + d_n \,, \quad \sigma_D^2 = \sigma_{d_1}^2 + \sigma_{d_2}^2 + \dots + \sigma_{d_n}^2$$

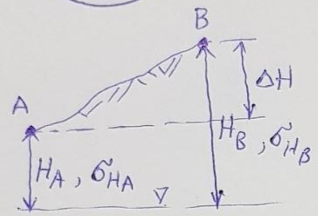
②



$$P = 2\pi R$$

$$\sigma_P^2 = (2\pi)^2 \cdot \sigma_R^2 \Rightarrow \sigma_P = 2\pi \cdot \sigma_R$$

③

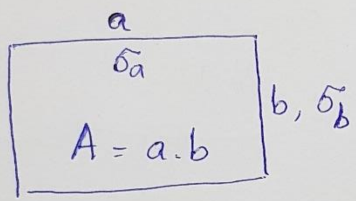


$$\Delta H = H_B - H_A$$

$$\sigma_{\Delta H}^2 = (1)^2 \cdot \sigma_{H_B}^2 + (-1)^2 \cdot \sigma_{H_A}^2$$

$$\Rightarrow \sigma_{\Delta H}^2 = \sigma_{H_B}^2 + \sigma_{H_A}^2$$

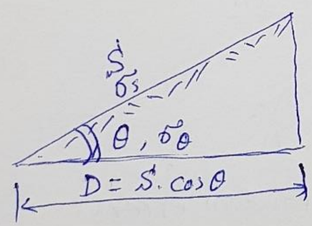
④



$$A = a \cdot b$$

$$\sigma_A^2 = b^2 \cdot \sigma_a^2 + a^2 \cdot \sigma_b^2$$

⑤



$$D = S \cdot \cos\theta$$

$$\sigma_D^2 = \cos^2\theta \cdot \sigma_S^2 + (-S \cdot \sin\theta)^2 \cdot \sigma_\theta^2$$

should be in radian

If $\sigma_\theta = \pm 5''$ $\Rightarrow$ multiply by $\frac{1}{3600} \cdot \frac{\pi}{180}$

EXAMPLE:

The radius (r) of a circular tract of land is measured to be 40.25 m with an estimated standard error ($\hat{\sigma}_r$) of ±0.01 m. Compute the area (A) of the tract of land and its estimated standard error ($\hat{\sigma}_A$).

SOLUTION:

$$A = \pi r^2 = \pi(40.25)^2 = 5089.58 \text{ m}^2$$

By the law of propagation of random errors:

$$\hat{\sigma}_A{}^2 = \left(\frac{\partial A}{\partial r}\right)^2 \hat{\sigma}_r{}^2 \Rightarrow \hat{\sigma}_A = \left(\frac{\partial A}{\partial r}\right)\hat{\sigma}_r$$

$$\hat{\sigma}_A = (2\pi r)\hat{\sigma}_r = \pm(2\pi \text{ x } 40.25)(0.01) = \pm 2.53 \text{ m}^2$$

$\Rightarrow$ A = 5089.58 ± 2.53 m^2

EXAMPLE:

Two sides and the included angle of a triangular land parcel were measured with the following results: a = 45.12 ± 0.05 m, b = 38.64 ± 0.03 m, and θ = 52° 15′ ± 30″. Calculate the area of the land parcel and its standard error.

SOLUTION:

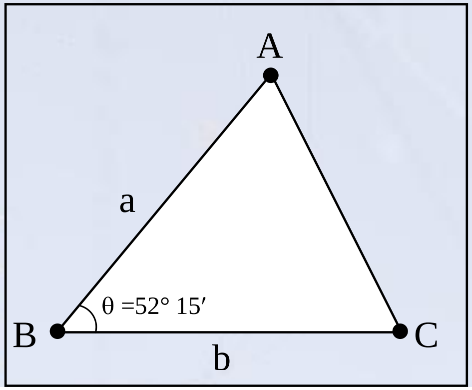


FIGURE 2.7: A triangular land parcel.

The area of the triangle is given by the following relationship:

$$A = \frac{1}{2} ab \sin\theta = \frac{1}{2} \text{ x } 45.12 \text{ x } 38.64 \sin(52^\circ\ 15') = 689.26 \text{ m}^2$$

The standard error of the area is (from Equation 2.12):

$$\hat{\sigma}_A = \pm\sqrt{\left(\frac{\partial A}{\partial a}\right)^2 \hat{\sigma}_a^{\ 2} + \left(\frac{\partial A}{\partial b}\right)^2 \hat{\sigma}_b^{\ 2} + \left(\frac{\partial A}{\partial \theta}\right)^2 \hat{\sigma}_\theta^{\ 2}}$$

$$\hat{\sigma}_a = \pm 0.05 \text{ m}, \quad \hat{\sigma}_b = \pm 0.03 \text{ m}, \quad \hat{\sigma}_\theta = \pm \frac{30}{3600} \text{ x } \frac{\pi}{180} = 1.454\text{x}10^{-4} \text{ radian}$$

$$\frac{\partial A}{\partial a} = \frac{1}{2} b \sin\theta = \frac{1}{2} \text{ x } 38.64 \sin(52^\circ\ 15') = 15.28 \text{ m}$$

$$\frac{\partial A}{\partial b} = \frac{1}{2} a \sin\theta = \frac{1}{2} \text{ x } 45.12 \sin(52^\circ\ 15') = 17.84 \text{ m}$$

$$\frac{\partial A}{\partial \theta} = \frac{1}{2} ab \cos\theta = \frac{1}{2} \text{ x } 45.12 \text{ x } 38.64 \cos(52^\circ\ 15') = 533.68 \text{ m}^2$$

$$\Rightarrow \hat{\sigma}_A = \pm\sqrt{(15.28)^2(0.05)^2 + (17.84)^2(0.03)^2 + (533.68)^2(1.454\text{x}10^{-4})^2} = \pm 0.94 \text{ m}^2$$

2.9 WEIGHTS AND WEIGHTED MEAN

- Simple Mean: measurements are done with the same degree of care.

- Weighted Mean: some measurements are more reliable than others.

- Weight: the degree of reliability of a measurement.

$D_A = 60.64 \pm 0.04\, m$, $D_B = 60.56 \pm 0.02\, m$

$w_i \propto \frac{1}{\sigma_i^2} \Rightarrow w_i = \frac{k}{\sigma_i^2}$

$$\hat{X} = \frac{w_1 \cdot x_1 + w_2 x_2 + \dots + w_n x_n}{w_1 + w_2 + \dots + w_n} = \frac{\frac{k}{\sigma_1^2} x_1 + \frac{k}{\sigma_2^2} x_2 + \dots + \frac{k}{\sigma_n^2} x_n}{\frac{k}{\sigma_1^2} + \frac{k}{\sigma_2^2} + \dots + \frac{k}{\sigma_n^2}}$$

$$= \frac{\frac{1}{\sigma_1^2} x_1 + \frac{1}{\sigma_2^2} \cdot x_2 + \dots + \frac{1}{\sigma_n^2} \cdot x_n}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \dots + \frac{1}{\sigma_n^2}}$$

(Value of k does not affect the result)

→ Let $k = 1$

$\Rightarrow w_1 = \frac{1}{(0.04)^2} = 625 = 625 * 1$

$w_2 = \frac{1}{(0.02)^2} = 2500 = 625 * 4$

Let $k = (0.04)^2$

$\Rightarrow w_1 = \frac{(0.04)^2}{(0.04)^2} = 1$

$w_2 = \frac{(0.04)^2}{(0.02)^2} = 4$

In general, take $k = \sigma_0^2 = (\text{Largest } \sigma)^2$

σ_0 is called standard error of unit weight

~~~~ $w_i = \frac{\sigma_0^2}{\sigma_i^2}$

$$\sigma_{\hat{X}} = \pm \frac{\sigma_0}{\sqrt{\Sigma w_i}}$$
~~~~

EXAMPLE:

Compute the weighted mean ($\hat{\ell}$) and the estimated standard error of the weighted mean ($\sigma_{\hat{\ell}}$) for the following four independent measurements of a distance:

ℓ_1 = 2746.34 ±0.02 ft $\quad$ ℓ_2 = 2746.38 ± 0.06 ft

ℓ_3 = 2746.26 ± 0.05 ft $\quad$ ℓ_4 = 2746.31 ± 0.04 ft

SOLUTION:

Let σ_0 = ± 0.06 ft, then:

$$w_1 = \left(\frac{0.06}{0.02}\right)^2 = 9, \qquad w_3 = \left(\frac{0.06}{0.05}\right)^2 = 1.44$$

$$w_2 = \left(\frac{0.06}{0.06}\right)^2 = 1, \qquad w_4 = \left(\frac{0.06}{0.04}\right)^2 = 2.25$$

$$\hat{\ell} = \frac{2746.34 \text{ x } 9 + 2746.38 \text{ x } 1 + 2746.26 \text{ x } 1.44 + 2746.31 \text{ x } 2.25}{9 + 1 + 1.44 + 2.25}$$

= 2746.33 ft

$$\sigma_{\hat{\ell}} = \pm \frac{0.06}{\sqrt{13.69}} = \pm 0.02 \text{ ft}$$

2.10 SIGNIFICANT FIGURES

- These are the digits with known values.
- **Examples:**

456.300	6 significant figures
0.0036	2 significant figures
6.000350	7 significant figures
54.0	3 significant figures
$2.1x10^{-5}$	2 significant figures

RULES:

- Any measured value should correspond with its standard error.
- The number of decimal places in a measurement should not exceed the accuracy of the fieldwork.
- When performing addition, subtraction, multiplication or division, the answer can not be more precise than the least precise number included in the mathematical operation. For example:

$$
\begin{array}{r}
24.217 \\
+\ 468.46 \\
+1563.1 \\
\hline
2055.777
\end{array}
$$

The sum must be rounded off to 2055.8 because 1563.1 has only one decimal place.