Rate of Return Computations
For a single Project

Rate of return (ROR) is the rate of interest paid on the unpaid balance of borrowed money, or the rate of interest earned on the unrecovered balance of an investment, so that the final payment or receipt brings the balance to exactly zero with interest considered.

The ROR is expressed as a percent per period, for example, I = 10% per year.
-100% < i < &
I = 100% means the entire amount is lost.

The definition above does not state that the rate of return is on the initial amount of the investment, but it is rather on the unrecovered balance, which varies with time. The following ex. Illustrate the difference between these concepts.

Ex:
A_t i = 10% per year, 1000$ investment is expected to produce a net cash flow of 315.47$ for each of 4 years.

$$A = 1000 \ (A/P, 10\%, 4) = 315.47\$$$

This represents a 10% per year rate of return on the unrecovered balance.
Compute the amount of the unrecovered investment for each of the 4 years using:

a) the rate of return on the unrecovered balance.
b) the rate of return on the initial 1000$ investment.

solution:

a) Table- 1 presents the unrecovered balance for each year after 4 years the total 1000$ investment is recovered and the balance in column A is exactly zero.

b) table (2) shows the unrecovered balance if the 10% return is always on the initial investment of 1000$. Column (6) in year 4 shows a remaining unrecovered amount of 138.12$, because only 861.88$ is recovered in the 4 years.

Table – 1

Year	Beginning unrecovered balance	Interest on unrecovered balance = 0.1	Cash Flow	Recovered amount Cop. 4-3	Ending unrecovered balance (9) + (5)
0	-	-	-1000$	-	-1000$
1	-1000$	100	315.47	215.47	-784.53
2	-784.53	78.45	315.47	237.02	-547.51
3	-547.51	54.75	315.47	260.72	-286.79
4	-286.79	28.68	315.47	286.79	0
		261.88$		1000$	

Table – 2

Year	Beginning unrecovered balance	Interest on unrecovered balance = 0.1	Cash Flow	Recovered amount Cop. 4-3	Ending unrecovered balance (9)+ (5)
0	-	-	-1000$	-	-1000$
1	-1000$	100$	315.47	215.47	-784.53
2	-784.53	100$	315.47	215.47	-569.06
3	-569.06	100$	315.47	215.47	-353.59
4	-353.59	100$	315.47	215.47	-138.12
		400$		861.88$	

To determine the rate of return I of a project's cash slow, set up the ROR relation.

The present worth of investments or disbursement PW_D is equated to the present worth of income or receipts, PW_R

$PW_D = PW_R$

$O = -PW_D + PW_R$

The annual-worth approach utilizes the AW values in the same fashion to solve for i

$$AW_D = AW_R$$
$$O = -AW_D + AW_R$$

ROR calculations using a Present worth:

In rate- of – return calculations, the objective is to find the interest rate i* at which the present sum and future sum are equivalent. The calculation made here are the reverse of calculations made in previous.

Ex:

If you deposit 1000$ now and are promised payments of 500$ three years from now and 1500$ five years from now find the ROR.

Solution:

The ROR relation using PW is

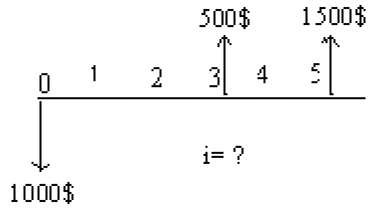
1000 = 500 (P/F1 i*, 3) + 1500 (P/F, i*, 5)

Where the value of i* to make the equality correct is to be computed. If the 1000$ is moved to the right side of eq., we have

O = - 1000 + 500 (P/F, i*, 3) + 1500 (P/F, i*, 5)

The equation is solved for i to obtain i* = 16.9%

Moreover, the rate of return will always be greater than zero if the total amount of receipts is greater that the total amount of disbursements, when, the time value of money is considered.



There are two common ways to determine i* once the PW relation is established:

- manual solution via trial and error
- Computer solution via spread sheet.

Using manual trial and error method to determine i*, If the cash flows are combined in such a manner that the income and disbursements can be represented by a single factor such as P/F or P/A, it is possible to look up the interest rate (in the tables) corresponding to the value of that factor for n years.

The problem, then, is to combine the cash flows into the format of only one of the standard factors. This may be done through the following procedure:

1- Convert all disbursements into either single amounts (P or F) or uniform amounts (A) by neglecting the time value of money.

For example

1- if it is desired to convert an A into an F value, simply multiply the A by the number of years.
2- Convert all receipts to either single or uniform values.
3- Having combined the disbursements and receipts so that either a P/F, P/A or A/F format applies, use the interest tables to find the approximate interest rate at which the P/F, P/A or A/F value, respectively, is satisfied for the proper n value.

It is important to recognize that the rate of return obtained in this manner is only an estimate of the actual rate of return, because the time value of money is neglected.

Ex:

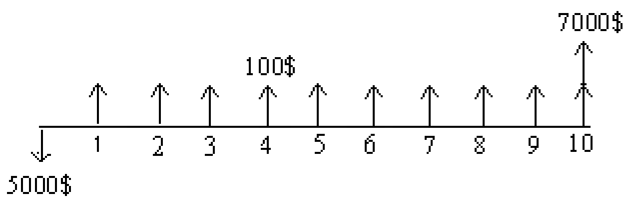
If 5000$ is invested now in common stock that is expected to yield 100$ per year for 10 years and 7000$ at the end of 10 years, what is the rate of return?

Solution:

Use the manual trial and error procedure based on a PW.

Equation:

1- the cash flow diagram



2- Use eq. format

O = -5000+100 (P/A, i*, 10) + (P/F, i*, 10).

3- Use the interest – rate estimation

The P/F factor is selected because most of the cash flow (i.e. 7000$) already fits this factor and errors created by neglecting the time value of the remaining money will be minimized.

P = 5000$, n = 10, and F = 10 (100) + 7000 = 8000

Now, we can state that:

5000 = 8000 (P/F, I, 10)

(P/F, I, 10) = 0.625

The approximate i is between 4% and 5%. Threfore, use i = 5% to estimate the actual rate of return.

0 = - 5000+100 (P/A, 5%, 10) + 7000 (P/F, 5%, 10)

0 **<** 69.46$

We are too large on the positive side, indicating that the return is more than 5%. Try i = 6%

0 = -5000+100 (P/A, 6%, 10)+ 7000 (P/F, 6%, 10)

0 **<** -355.19$

Since the interest rate of 6% is too high, interpolate between 5% and 6% to obtain

$$I = 5.00 + \frac{69.46-0}{69.46-(-355.19)}$$

$$= 5 + 0.16 = 5.16\%$$

The second method by using Excel spreadsheet systems.

Rate of Return calculations using an annual worth eq.:

The procedure is as follows:

1- Draw a cash-flow diagram.
2- Set up the relations for the AW_D and AW_R
3- Set up the RoR relation in the form eq.
   $0 = - AW_D + AW_R$
4- Select values of I by trial and error until the equation is balanced. If necessary interpolate to determine it.

Section 7.5

Multiple Rate of Return Values:

In previous a unique i* value was determined for the given cash-flow sequences. Investigation shows that the signs on the cash flows changed only once, usually from minus in year 0 to plus for the rest of the investment's life. This is called a conventional cash flow.

If there is more than one sign change, the series is called nonconventional.

Examples:

Type	**Sign of cash flow**							**Number of sign changes**
	0	**1**	**2**	**3**	**4**	**5**	**6**	
conventional	-	+	+	+	+	+	+	1
conventional	-	-	-	+	+	+	+	1
Conventional	+	+	+	+	-	-	-	1
Non conventional	-	+	+	+	-	-	-	2
Non conventional	+	+	-	-	-	+	+	2
Non conventional	-	+	-	-	+	+	+	3

The total number of i* values is less than or equal to the number of sign changes in the sequence.

Ex:

Plot the present worth versus the rate of return for i values of 5, 10, 20, 30, 40 and 45% for the following cash flows:

Year	0	1	2	3
Cash flow	+2000	-500	-8100	+6800

<u>Internal and composite Rates of Return:</u>

The rat of return values we have computed thus far have assumed that any net positive cash flows (receipts) are reinvested immediately at the rate of return that balances the rate of return equation.

The rate, which is computed by eq. * ↑ * ↑ is called the internal rate of return because it does not consider any of the economic factors external to the project.

The internal rate of return i^* is an interest rate of a project which assumes that all positive cash flows are reinvested at a rate of return that balances the rate of return equation.

The reinvestment rate, symbolize by C, is often set equal to the MARR. The interest rate determined in this fashion to satisfy the rate of return equation will be called the composite rate of return and will be symbolized by i'. By definition:

The composite rate of return i' is the interest rate of a project which assumes that net positive cash flows which represent funds not immediately needed in the project are reinvested at the rate C, which is explicitly stated and has been determined by considering factors external to the project cash flow.

The term composite is used for i' because it is determined conditional upon the reinvestment rate C. (If C happens to equal any one of the i^* values, then i' will equal that i^* value).

The reinvestment rate is applied to net positive cash flows.

The i´ value must cause the net overall investment for the project itself to be exactly zero at the end of the project.

The project net investment technique:

For each year find the future worth F of the project net investment one year in the future, that is, F_{t+1} using F_t and the cash flow in year t, C_t.

The interest rate in the F/P factor is C if the net investment F_t is positive and it is i´ if F_t is negative. Mathematically, for each year set up the relation

$$F_0 = C_0$$
$$F_{t+1} = F_t\,(F/P_{1i\%}\,,\,1) + C_{t+1} = F_t\,(1+i) + C_{t+1}$$

Where n – total years in the project.

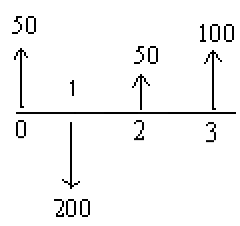
$$I = \begin{cases} C \text{ if } Ft > 0 \text{ (net positive investment)} \\ i' \text{ if } Ft < 0 \text{ (net negative investment)} \end{cases}$$

Form the relation Fn = 0 and use trial and error to find the unique i´ value. The rate C is used in eq. when the project investment has been recovered and excess cash flow returns the reinvestment rate.

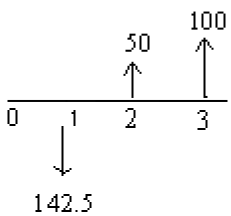
Ex.:

The development of F_0 through F_s for the cash flow sequence below using a 15% reinvestment rate is as follows:

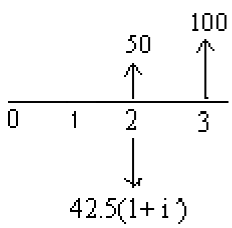
Cash flow	0	1	2	3
	50	-200	50	100
Cumulative Cash flow, $	50	-15	-100	0



(a)



(b)



The net investment for year t = 0 is $F_0 = C_0 = 50$.

In year t = 1, the 50$ returns C = 15% since $F_0 > 0$ and for t = 1

$$F_1 = 50\ (1 + 0.15) - 200 = -142.5\$$$

Since the project value is negative in t = 1, the value F_t earns at the rate i´ for the next year.

$$F_2 = -142.5\ (1+ i') + 50$$

Since this result must be negative for all i´ > 0, use i´ to find F_3

$$F_3 = F_2\,(1 + i') + C_3 = (-142.5)\ (1+ i') + 50)\ (1+ i') + 100$$

Setting eq. * ↑ equal to zero and solving for i´ will result in the unique composite rate of return for the cash-flow sequence.

This project net-investment procedure to fine i´ may be summarized as follows:

1- Draw a cash-flow diagram of the original cash flow.
2- Develop the series of project net investment using eq. ↑ (F_{t+1}) and the stated C value. The results is the F_n expression in terms of i´.
3- Set the F_n expression equal to 0 and find i´ value to balance the equation. If necessary interpolate to determine i´.

If the reinvestment rate C equals i´, the value i´ will be the same as i*.

It is possible to summarize the relations between C, i´ and any i* as follows:

Relation between Reinvestment (C) and IRR (i*)	**Relation between (i´) and IRR (i*)**
C = i*	i´ = i*
C < i*	i´ < i*
C > i*	i´ > i*