Nominal and Effective Interest Rates and
Continuous compounding

Nominal and effective interest rates are used when the compounding period (or interest period) is less than one year.

Nominal interest rate, r, equal

r = interest rate per period x number of periods

For example, a period interest rate listed as 15% per month could also be expressed as a nominal 45% per quarter (that is, 15% per month x 3 months per quarter), 9% per semiannual period, 18 per year. The nominal interest rate obviously ignores the time value of money and the frequency with which interest is compounded.

When the time value of money is taken into consideration in calculating interest rates from period interest rates, the rate is called an effective interest rate.

The frequency of the payments or receipts is known as the payment period (pp).

For example, if a company deposited money each month into an account that pays a nominal interest rate of 14% per year compounded semiannually, the payment period would be 1 month while the compounding period would be 6 months. Similarly, if a person deposits money each year into a savings account which compounds interest quarterly, the payment period 1 year, while the compounding period is 3 months.

Effective interest rate formulation:

To illustrate the difference between nominal and effective interest rates, the future month of 100$ after 1 year is determined using both rates.

If a bank pays 12% interest compounded annually, the future worth of 100$ is:

$$\mathbf{F = P\ (1+i)^n = 100\ (1.12)^1 = 112\$}$$

If the bank pays interest that is compounded semiannually, the future worth must include the interest on the interest earned in the first period. An interest rate 12% per year compounded semiannual means that the bank will pay 6% interest after 6 months and another 6% after 12 months.

Future worth values of 100$ after 6 months and after 12 months are

$$F_6 = 100\ (1+0.06) = 106\$$$
$$F_{12} = 105\ (1+0.06) = 112.36\$$$

Therefore, the effective annual interest rate is 12.36%.

The equation to determine the effective interest rate from the nominal interest rate may be generalized as follows:

$$i = \left(1 + \frac{r}{m}\right)^m - 1$$

Where

i – effective interest rate per period
r – nominal interest rate per period
m – number of compounding periods

Example:

A national credit card carries an interest rate of 2% per month on the unpaid balance:

a) calculate the effective rate per semiannual period.
b) If the interest rate is stated as 5% per quarter, find the effective rates per semiannual and annual time periods.

Solution:

(a)

r = 2% per month x 6 months per semiannual period
= 12% per semiannual period

The min. eq.* is equal to 6, since interest would be compounded 6 times in a 6 month time period. Thus the effective semiannual rate is:

$$i \text{ per 6 months} = \left(1 + \frac{0.12}{6}\right)^6 - 1$$

= 0.1262 (12.62%)

(b) For an interest rate of 5% per quarter, the compounding period is quarterly:

m = 2, and r = 10%, thus

$$i \text{ per 6 months} = \left(1 + \frac{0.10}{2}\right)^2 - 1 =$$

= 0.1025 (10.25%)

The effective interest rate per year can be determined using r = 20% and m = 4 as follows:

$$i \text{ per year} = \left(1 + \frac{0.2}{4}\right)^4 - 1 = 0.2155 \text{ (21.55\%)}$$

Ex:

A university credit union advertises that its interest rate on loans is 1% per month, calculate the effective annual interest rate and use the interest factor tables to find the corresponding P/F factor for n = 8 years.

Solution:

r per year = (0.01) (12) = 0.12 and m = 12

$$i = \left(1+ \frac{0.12}{12}\right)^{12} - 1 =$$

$$= 1.1268 – 1 = 0.1268\ (12.68\%)$$

From table (P/F, 12.68%, 8) = 0.3858

<u>Effective interest rates for continuous compounding</u>:

M – infinity

$$\boxed{I = e^{r} - 1}$$

eq.** is used to complete the effective continuous interest rate.

As an illustration, for an annual nominal rate of 15% per year (r = 15% per year), the effective continuous rate per year is

$$i = e^{0.15} – 1 = 0.16183\ (16.183\%)$$

<u>Ex:</u>

a) For an interest rate of 18% per year compounded continuously, calculate the effective monthly and annual interest rates.

Solution:

The nominal monthly rate r is $\frac{18}{12}$ = 1.5% = 0.015 per month.

i per month = $e^{r} – 1 = e^{0.015} – 1 = 0.01511$ (10511%) similarly, the effective annual rate is r = 0.18

i per year = $e^{r} – 1 = e^{0.18} – 1 = 0.1972$ (19.72%)

calculations for payment periods equal to or longer than compounding periods:

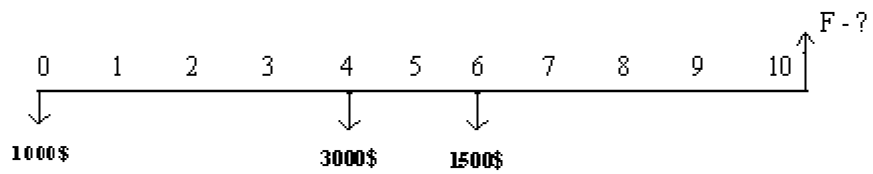
P = F (P/F, effective i per period, number of periods)
F = P (F/P, effective i per period, number periods)

Effective interest rate, i	**Units for n**
1% per month	Months
3% per quarter	Quarters
6% per 6 months	Semiannual periods
12% per year	Years
24% per 2 years	2-year period

Ex:

If a woman deposits 1000$ now, 3000$ four years from now, and 1500$ six years from now at an interest rate of 12% per year compounded semiannually, how much money will she have in her account 10 years from now?

Solution:



i per year = $(1 + \frac{0.12}{2})^2 - 1 = 0.1236$ (12.36%) since I has units of per year, n must expressed in years, thus

F = 1000 (F/P, 12.36%, 10) + 3000 (F/P, 12.36%, 6) + 1500 (F/P, 12.36%, 4) = 11.634.5$

Alternatively, are can use the effective rate of 6% per semiannual period and then use semiannual periods for n. In this case, the future worth is

F = 1000 (F/P, 6%, 20) + 3000 (F/P, 6%, 12) + 1500 (F/P, 6%, 8) = 11.634.5$

Uniform – series and Gradient factors:

The relationship between the compounding period CP, and payment period PP must be determined. The relationship will be one of the following three cases:

Case 1. the payment period is equal to the compounding period PP= CP.
Case 2. The payment period is longer than the compounding period PP>CP.
Case 3. The payment period is shorter than the compounding period, PP<CP.

For either case 1 or case 2, where PP = CP or PP>CP, the following procedure always applies:

Step 1. count the number of payments and use that number as n. For example, if payments are made quarterly for 5 years, n is 20 quarters.

Step 2. find the effective interest rate over the same time period as n in step. 1, for example if n is expressed in quarter, then the effective interest rate per quarter must be found.

Step 3. Use these values of n and i in the standard notation equations or formulas.

Examples of n and I values where PP=CP or PP>CP

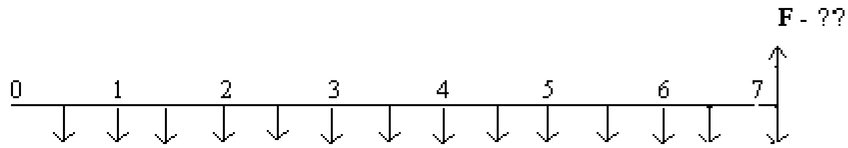
Cash. Flow sequence	Interest Rate	What to find What is given	Standard notation
500$ semiannually for 5 years	**16% per year compounded semiannually**	Find P1 given A	P = 500 (P/A, 8%, 10)
75$ monthly for 3 years	24% per year compounded monthly	Fine F, given A	F = 75 (F/A, 2%, 36)
25$ per month increase for 4 years	1% per month	Find P, given C	P = 25 (p/c, 1%, 48)
5000$ per quarter for 6 years	1% per month	Find A, given P.	A = 5000 (A/P, 3.03%, 24)

Example:

If a woman deposits 500% every 6 months for 7 years, how much money will she have in her investment portfolio after the last deposit if the interest rate is 20% per year compounded quarterly?

Solution:

The cash flow diagram is shown in fig.



C = 20% per year compounded quarterly

The compounding period (quarterly) is shorter than the payment period (semiannual), the number of payments, is 14,

the future worth is F = 500 (F/A, i, 14) since n is expressed in semiannual periods, an effective semiannual interest rate, with r = 0.1 per 6 month

$$I \text{ per 6 moths} = (1 + 0.1)^2 - 1 =$$
$$= 0.1025 \quad (10.25\%)$$
$$F = A\ (F/A, 1025\%, 14) =$$
$$= 500\ (28.4891) =$$
$$= 14, 244.5\$$$

Calculations for payment periods shorter than compounding periods: (PP<CP).

The procedure to calculate the future worth or present worth depends on the conditions specified regarding interperiod compounding.

There are three possible scenarios:

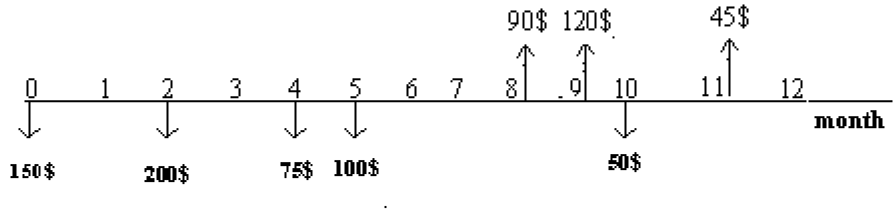
1- There is no interest paid on the money deposited (or withdrawn) between compounding periods.
2- The money deposited (or withdrawn) between compounding periods earns simple interest.
3- All interperiod transactions earn compound interest.

Only scenario 1 (no interest on interperiod transactions) will be considered here, since most real-word transactions fall into this category.

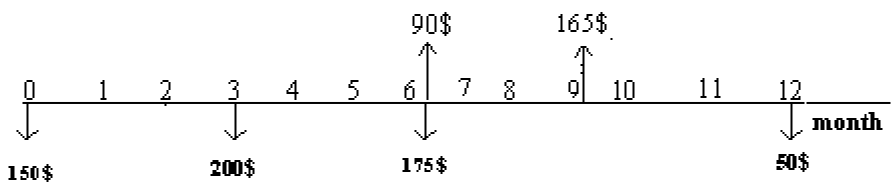
If no interest is to be paid on interperiod transaction then any amount of money that is deposited or withdrawn between compounding periods is regarded as <u>having been deposited at the end of the compounding period or withdrawn at the beginning of the compounding period</u>. (this is the usual mode of operation of banks and other institutions).

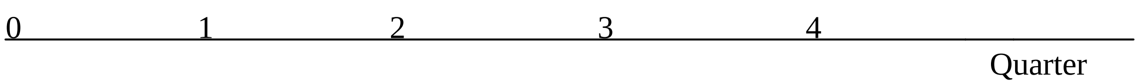
Thus, if the compounding period were a quarter, the actual transactions shown in fig. (a) would be treated as shown in fig. (b). To find the present worth of the cash flow represented by fig. (b), the nominal yearly interest rate is divided by 4 (since interest is compounded quarterly) and the appropriate n value is used in the P/F or F/P factor. For example, the interest rate is 12% compounded quarterly for the fig. cash flow.

$$P = -150 - 200\ (P/F, 36, 1) - 85\ (P/F, 3\%, 2) + 165\ (P/F, 3\%, 3) -$$
$$-50\ (P/F, 3\%, 4) =$$
$$= -317.73\$$$



(a)





(b) Diagram of cash flows for quarterly compounding periods using no.

Ex.:

Company plans to place money in a project of deposit that pays 18% per year compounded daily. What effective rate will company receive:

a) yearly and b) semiannually

Solution:

R = 0.18 and m = 365

i per year $\left(1+\frac{0.18}{365}\right)^{365} - 1 = 0.19716$ (19.71%)

That is, company will get an effective 19.71% per year on it deposit.

b) r = 0.09 per 6 months and m = 182 days

$$i \text{ per 6 months} = \left(1+\frac{0.09}{182}\right)^{182} - 1 =$$

$$= 0.09415\ (9.415\%)$$

Ex.2

Two companies both plan to invest 5000$ for 10 years at 10% per year. Compute the future worth for both individuals if company (1) gets interest compounded annually and company (2) gets continuous compounding.

Solution:

Company (1) F = P/F/P, 10%, 10) = 5000 (2.5937) = 12.96981

Company (2) using the continuous – compounding: the effective i per year

$$i = e^{0.1} - 1 = 0.10517\ (10.517\%)$$

The future worth is

F = P (F/P, 10.517%, 10) = 5000 (2-7183) = ?????????????????

Ex.:

If 2000$ is deposited each year for 10 years at an interest rate of 10% per year, compare the present worth for a) annual

b) continuous compounding

Solution:

a) For annual compounding

P = 2000 (P/A, 10%, 10) = 12.289$

b) For continuous compounding

$$i \text{ per year} = e^{0.1} - 1 = 0.10517$$

P = 2000 (P/A, 10.517%, 10) =

= 2000 (6.0104) = 12.021$

Ex:

Company wants to purchase a car for 8500$, the company plans to borrow the money from credit union and to repay it monthly over a period of 4 years. If the nominal interest rate is 12% per year compounded monthly, what will company monthly installments be?

Solution:

The compounding period equal the payment period (case 1), with an effective monthly rate of I = 1% per month and n = 12 (4) = 48 payments. Therefore, the monthly payments are

A = 8500 (A/P, 1%, 48) = 8500 (0.02633) =

= 223.84$