Investment Portfolio Management Course

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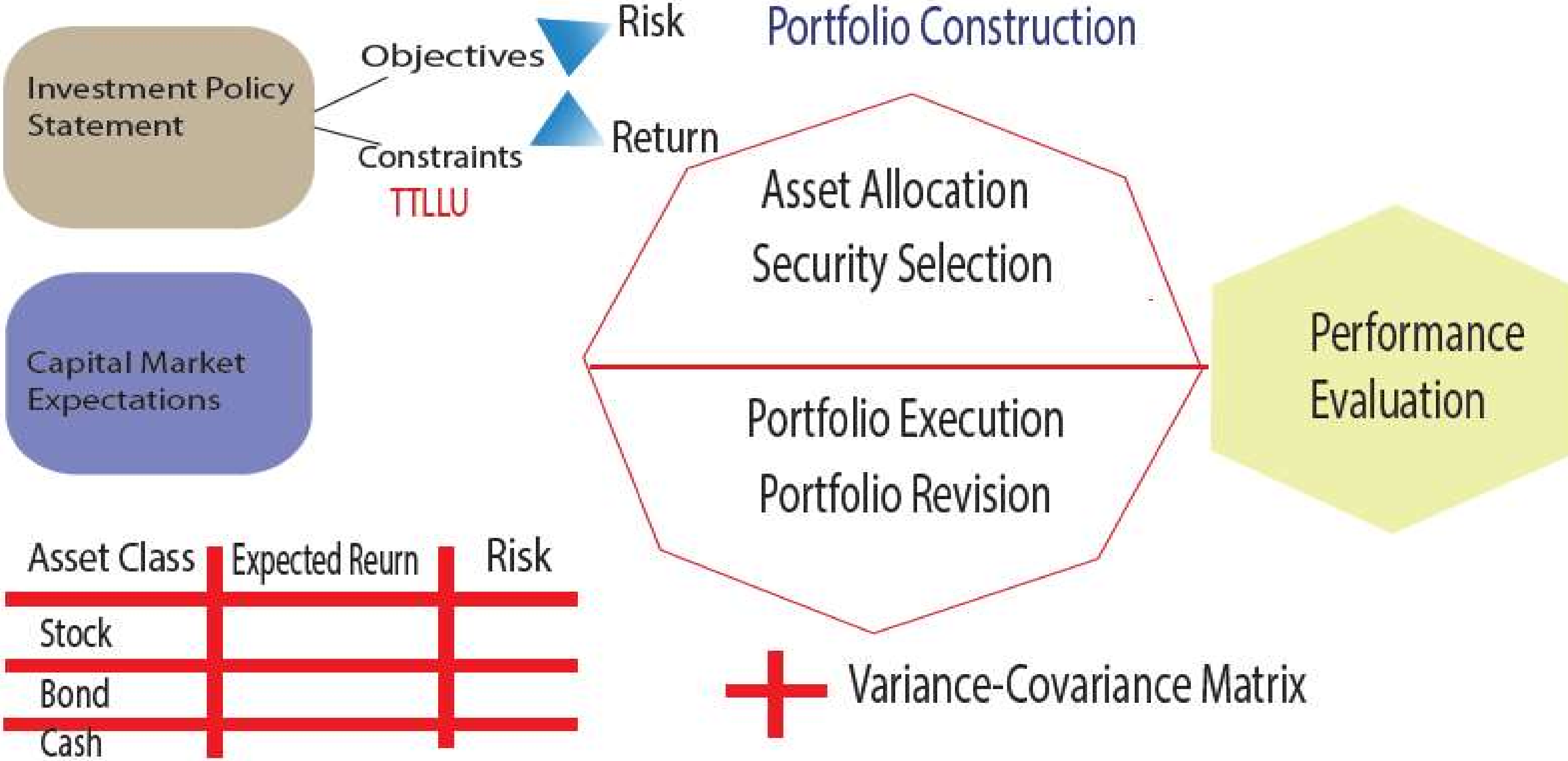
Summer Semester

An-Najah National University

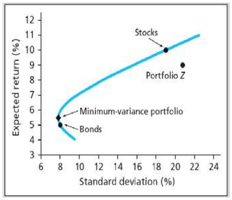
Chapter 6: Efficient Diversification

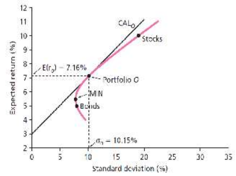
Portfolio Management Process

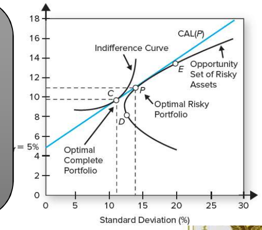


Summary of Portfolio Construction Process

Portfolio Construction Process	Objective	Process
A. Asset Allocation	Determine the **Optimal Risky Portfolio (ORP)**	1. Determine the Efficient Frontier of Risky Assets; 2. Determine the Optimal Risky Portfolio (P^o)
B. Capital Allocation	Determine The Preferred **Complete Portfolio**	Determine the optimal Allocation between the ORP and the Risk Free Asset







PORTFOLIO MANAGEMENT TOOLKIT

Introduction

- This Chapter Aims to Show how investors can construct the best possible risky portfolio with the help of efficient diversification through:
  - Measuring the return and risk of a portfolio.
  - Explaining why the covariance terms dominate portfolio risk.
  - Discussing the concept of an efficient portfolio.
  - Explaining how the optimal portfolio of risky securities is selected.
  - Clarifying the import of the separation theorem.
  - Showing how the single index model helps in obtaining the inputs required for applying the Markowitz model.

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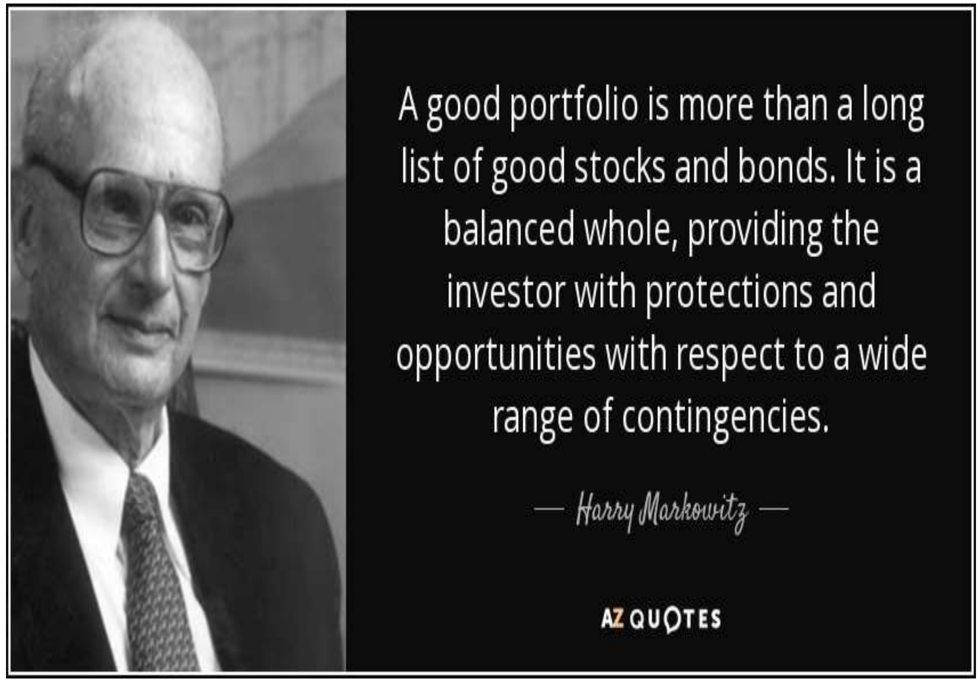
Introduction

- **Portfolio theory**, originally proposed by Harry Markowitz in the 1950s, was the first formal attempt to **quantify the risk of a portfolio** and develop a methodology for **determining the optimal portfolio.**



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Introduction

- **Harry Markowitz** was the first person to **show quantitatively** why and how **diversification** reduces **risk**.

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DIVERSIFICATION AND PORTFOLIO RISK

- What do we mean by **diversification**:

Spreading *an investment across a* ***number of assets*** *will eliminate some, but not all, of the risk.*

How diversification influences risk

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Example

- Suppose you have $**100,000** to invest and you want to invest it equally in **two stocks, A and B.** The return on these stocks depends on the state of the economy. Your assessment suggests that the **probability distributions** of the returns on stocks A and B are as shown below:

State of the Economy	*Probability*	*Return on Stock A*	*Return on Stock B*	*Return on Portfolio*
1	0.20	15%	–5%	5%
2	0.20	–5%	15	5%
3	0.20	5	25	15%
4	0.20	35	5	20%
5	0.20	25	35	30%

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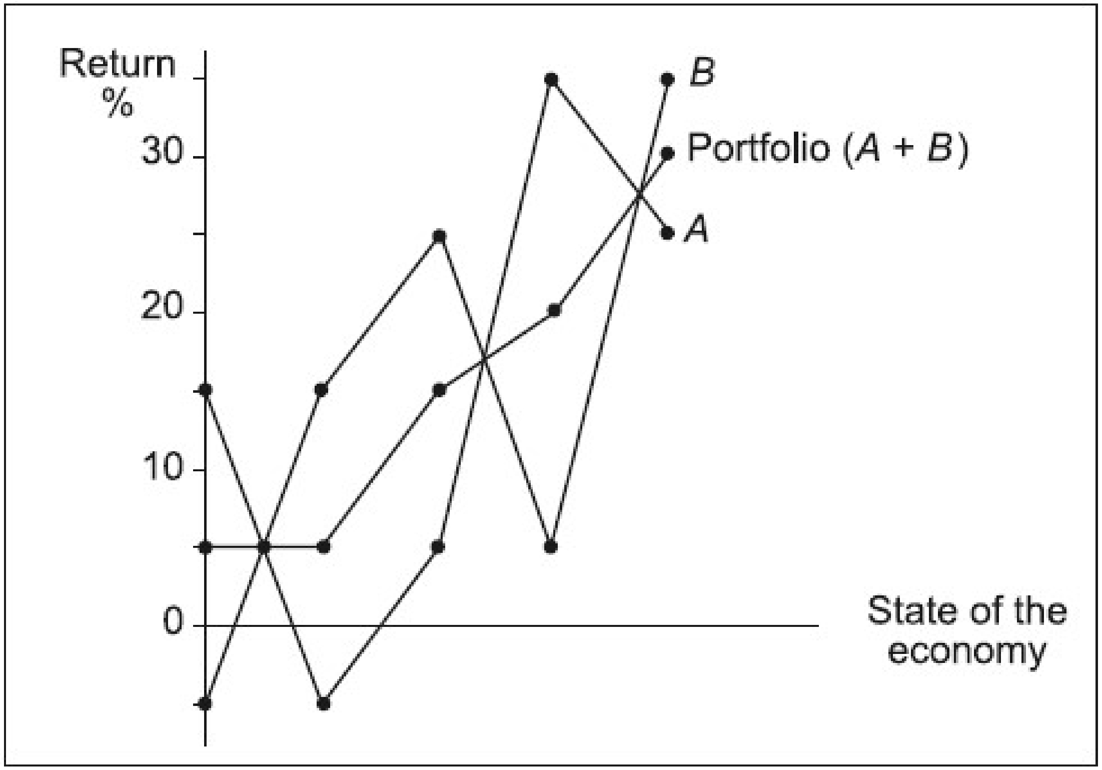
Returns on Individual Stocks and the Portfolio



EXPECTED RETURN OF THE PORTFOLIO

- The **expected return** and **standard deviation** of return on stocks A and B and the portfolio consisting of A and B in equal proportions are calculated using:

$$\underbrace{r^P}_{\text{Return of Portfolio}} = \underbrace{w^A}_{\text{Weight in Asset A}} \underbrace{r^A}_{\text{Return of Asset A}} + \underbrace{w^B}_{\text{Weight in Asset B}} \underbrace{r^B}_{\text{Return of Asset B}}$$

$$\underbrace{w^A}_{\text{Weight in Asset A}} = \frac{\text{Dollar Value of Asset A in Portfolio}}{\text{Total Dollar Value of Portfolio}}$$

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STANDARD DEVIATION OF THE PORTFOLIO

With 2 assets (N=2):

$$Var[r_p] = \sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$$

With N assets:

$$Var[r_p] = \sigma_p^2 = \sum_{i=1}^{N} w_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j>i}^{N} w_i w_j \,\mathrm{COV}(ri,rj)$$

$$\sigma_p = \sqrt{\sigma_p^2}$$

$$\rho_{1,2} = \frac{\mathrm{COV}(r1,r2)}{\sigma_1\sigma_2}$$

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STANDARD DEVIATION OF THE PORTFOLIO

		Expected Return
Stock A	:	0.2(15%) + 0.2(–5%) + 0.2(5%) + 0.2(35%) + 0.2(25%) = 15%
Stock B	:	0.2(–5%) + 0.2(15%) + 0.2(25%) + 0.2(5%) + 0.2(35%) = 15%
Portfolio of A and B	:	0.2(5%) + 0.2(5%) + 0.2(15%) + 0.2(20%) + 0.2(30%) = 15%

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STANDARD DEVIATION OF THE PORTFOLIO

Standard Deviation				
Stock A	:	σ^2_A	=	$0.2(15-15)^2 + 0.2(-5-15)^2 + 0.2(5-15)^2 + 0.2(35-15)^2 + 0.20\ (25-15)^2$
			=	200
		σ_A	=	$(200)^{1/2} = 14.14\%$
Stock B	:	σ^2_B	=	$0.2(-5-15)^2 + 0.2(15-15)^2 + 0.2(25-15)^2 + 0.2(5-15)^2 + 0.2\ (35-15)^2$
			=	200
		σ_B	=	$(200)^{1/2} = 14.14\%$
Portfolio	:	$\sigma^2_{(A+B)}$	=	$0.2(5-15)^2 + 0.2(5-15)^2 + 0.2(15-15)^2 + 0.2(20-15)^2 + 0.2(30-15)^2$
			=	90
		σ_{A+B}	=	$(90)^{1/2} = 9.49\%$

DIVERSIFICATION REDUCES RISK.

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STANDARD DEVIATION OF THE PORTFOLIO

- In general, if returns on securities **do not move in perfect lockstep**, **diversification reduces risk**.
  - In technical terms, **diversification reduces** risk if returns are **not perfectly positively correlated.**

- When a portfolio is made up of different stocks, the **variability** of the return will **depend on the correlation** between the various assets.

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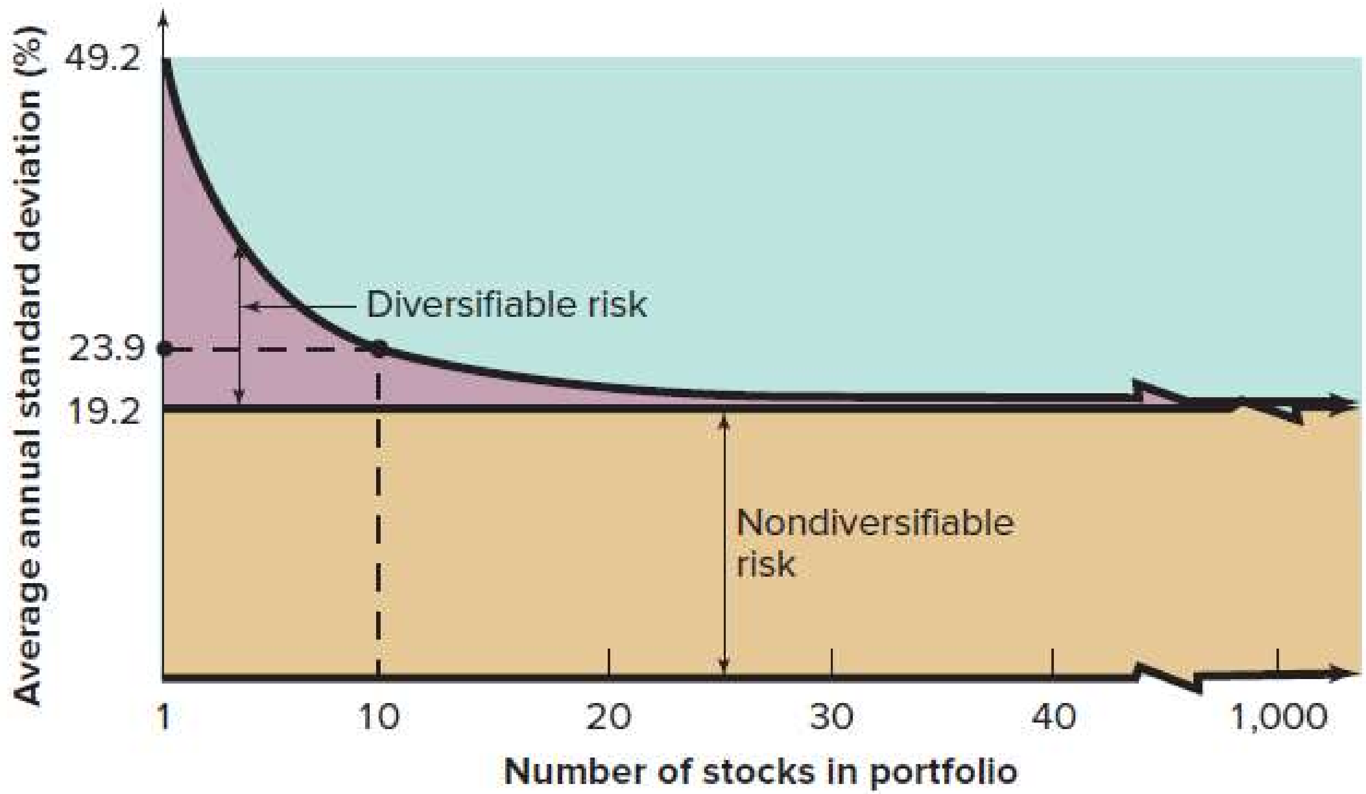
PORTFOLIO DIVERSIFICATION



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Market Risk Versus Unique Risk

- The **portfolio risk** does not fall below a certain level, irrespective of how wide the diversification is. **Why?**

$$\text{Total risk} = \text{Market risk} + \text{Unique risk}$$

Market Risk

- The **market risk** of a stock represents that portion of its risk which is attributable to **economy-wide factors** like the growth rate of GNP, the level of government spending, money supply, interest rate structure, inflation rate, wars and Corona virus,…etc.
  - These factors **affect all firms** to a greater or lesser degree, investors **cannot avoid the risk** arising from them, however diversified their portfolios may be.
  - It is also referred to as **systematic risk** (as it affects all securities) or **non-diversifiable risk**.

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Unique Risk

- The unique risk of a stock represents that portion of its total risk which **stems from firm-specific factors** like the development of a new product, a labor strike, or the emergence of a new competitor.
  - Events of this nature primarily **affect the specific firm** and not all firms in general.
  - The unique risk of a stock **can be eliminated** by combining it with other stocks.
  - In a diversified portfolio, unique risks of different stocks tend to cancel each other.
- **Unique risk** is also referred to as **firm specific risk**, **diversifiable risk**, **unsystematic risk** or **idiosyncratic risk**.

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RULES &
REGULATIONS

PORTFOLIO RETURN AND RISK

- while individual returns and risks are important, what matters finally is the **return and risk of the portfolio**.

$$E(R_p) = \sum_{i=1}^{n} w_i E(R_i)$$

$$\left(\sum_{i=1}^{n} w_i = 1 \right)$$

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PORTFOLIO RETURN AND RISK

- Just as **the risk of an individual security** is measured by the **variance** (or standard deviation) of its return, the **risk of a portfolio** too is measured by the **variance** (or standard deviation) of its return.

$$E(R_p) = \sum_{i=1}^{n} w_i \, E(R_i)$$

$$\sigma_p^{\,2} \neq \sum w_i^{\,2} \sigma_i^{\,2}$$

- Due to this inequality, investors can achieve the benefit of risk reduction through diversification.

MEASUREMENT OF COMOVEMENTS IN SECURITY RETURNS

- Calculating **portfolio risk** requires information on:
- 1) **Weighted** individual security **risks**, and
- 2) **Weighted comovements** between the returns of securities included in the portfolio.
  - Comovements between the returns of securities are **measured by covariance** (an absolute measure) and **coefficient of correlation** (a relative measure).

CALCULATION OF PORTFOLIO RISK

- **Portfolio Risk: 2-Security Case**
- The risk of a portfolio consisting of two securities is given by the following formula:

$$\sigma_p^{\ 2} = w_1^{\ 2}\,\sigma_1^{\ 2} + w_2^{\ 2}\,\sigma_2^{\ 2} + 2\,w_1 w_2\,\rho_{12}\,\sigma_1\,\sigma_2$$

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Coefficient of Correlation

$$\text{Cor}\,(R_i, R_j) \text{ or } \rho_{ij} = \frac{\text{Cov}\left(R_i, R_j\right)}{\sigma_i \,.\, \sigma_j} \text{ or } \frac{\sigma_{ij}}{\sigma_i . \sigma_j}$$

$$\sigma_{ij} = \rho_{ij} \,.\, \sigma_i \,.\, \sigma_j$$

- Increasingly positive correlation
  - Strong positive linear relationship (up to 1, which indicates a perfect linear relationship).
- Increasingly negative covariance
  - Strong negative (inverse) linear relationship (down to −1, which indicates a perfect inverse linear relationship).
- Zero correlation
  - No linear relationship.

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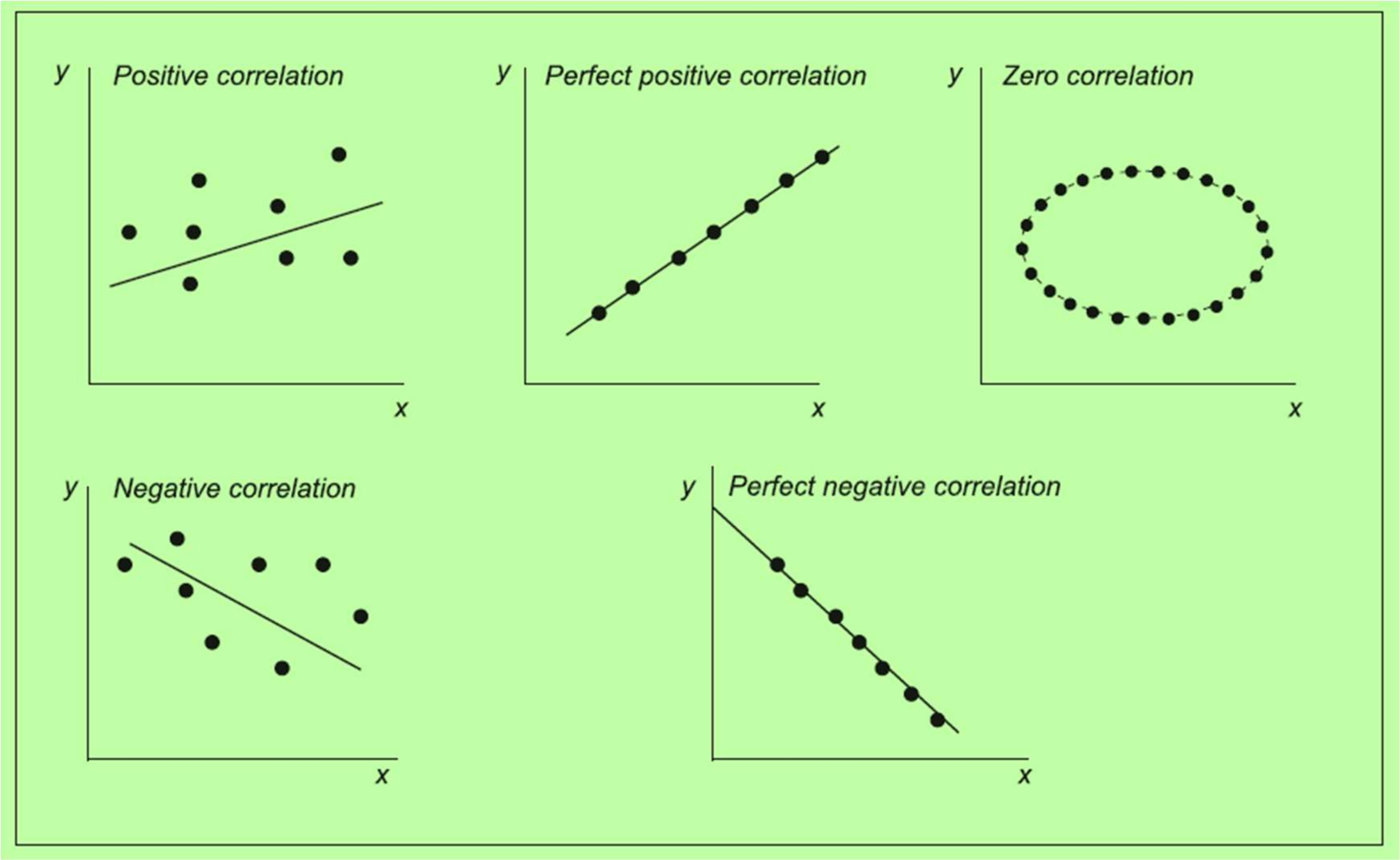
Various Types of Correlation Relationships



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MEASUREMENT OF COMOVEMENTS IN SECURITY RETURNS

- **Covariance** reflects the degree to which the returns of the two securities vary or change together.
  - A positive covariance means that the returns of the two securities move in the **same direction.**
  - A negative covariance implies that the returns of the two securities move in **opposite direction.**

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MEASUREMENT OF COMOVEMENTS IN SECURITY RETURNS

- The covariance between any two securities S and B is calculated as follows:

$$\mathrm{Cov}(r_{\mathrm{S}}, r_{\mathrm{B}}) = \sum_{i=1}^{S} p(i)[r_S(i) - E(r_S)][r_B(i) - E(r_B)]$$

Covariance can be:

- Positive: indicating a positive relationship,
- Negative: indicating an inverse relationship, or
- Zero: indicating the absence of a relationship between the variables

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EXAMPLE

	A	B	C	D	E	F
1			Stock Fund		Bond Fund	
2	Scenario	Probability	Rate of Return	Col B x Col C	Rate of Return	Col B x Col E
3	Severe recession	.05	−37	−1.9	−9	−0.45
4	Mild recession	.25	−11	−2.8	15	3.8
5	Normal growth	.40	14	5.6	8	3.2
6	Boom	.30	30	9.0	−5	−1.5
7	Expected or Mean Return:		SUM:	10.0	SUM:	5.0

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EXAMPLE

	A	B	C	D	E	F	G	H	I	J
1				Stock Fund			Bond Fund			
2				Deviation				Deviation		
3			Rate	from		Column B	Rate	from		Column B
4			of	Expected	Squared	×	of	Expected	Squared	×
5	Scenario	Prob.	Return	Return	Deviation	Column E	Return	Return	Deviation	Column I
6	Severe recession	.05	−37	−47	2209	110.45	−9	−14	196	9.80
7	Mild recession	.25	−11	−21	441	110.25	15	10	100	25.00
8	Normal growth	.40	14	4	16	6.40	8	3	9	3.60
9	Boom	.30	30	20	400	120.00	−5	−10	100	30.00
10				Variance = SUM		347.10			Variance:	68.40
11		Standard deviation = SQRT(Variance)				18.63			Std. Dev.:	8.27

$$Var(r) \equiv \sigma^2 = \sum_{s=1}^{S} p(s)[r(s) - E(r)]^2$$

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EXAMPLE

	A	B	C	D	E	F	G
1			Portfolio invested 40% in stock fund and 60% in bond fund				
2			Rate	Column B	Deviation from		Column B
3			of	×	Expected	Squared	×
4	Scenario	Probability	Return	Column C	Return	Deviation	Column F
5	Severe recession	.05	−20.2	−1.01	−27.2	739.84	36.99
6	Mild recession	.25	4.6	1.15	−2.4	5.76	1.44
7	Normal growth	.40	10.4	4.16	3.4	11.56	4.62
8	Boom	.30	9.0	2.70	2.0	4.00	1.20
9		Expected return:		7.00		Variance:	44.26
10					Standard deviation:		6.65

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EXAMPLE

	A	B	C	D	E	F
1			Deviation from Mean Return		Covariance	
2	Scenario	Probability	Stock Fund	Bond Fund	Product of Dev	Col B × Col E
3	Severe recession	.05	−47	−14	658	32.9
4	Mild recession	.25	−21	10	−210	−52.5
5	Normal growth	.40	4	3	12	4.8
6	Boom	.30	20	−10	−200	−60.0
7				Covariance =	SUM:	−74.8
8	Correlation coefficient = Covariance/(StdDev(stocks)*StdDev(bonds)) =					−0.49

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Asset Allocation with Two Risky Assets

- Three Rules
  - RoR: Weighted average of returns on components, with investment proportions as weights
  - E $r_P = w_B r_B + w_S r_S$ ected returns on components, with portfolio proportions as weights
  - Variance of RoR:

$$E(r_P) = w_B E(r_B) + w_S E(r_S)$$

$$\sigma_P^2 = (w_B \sigma_B)^2 + (w_S \sigma_S)^2 + 2(w_B \sigma_B)(w_S \sigma_S)\rho_{BS}$$

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Asset Allocation with Two Risky Assets

- Risk-Return Trade-Off
  - Investment opportunity set
    - Available portfolio risk-return combinations
- Mean-Variance Criterion
  - If $E(r_A) \geq E(r_B)$ and $\sigma_A \leq \sigma_B$
    - Portfolio *A* dominates portfolio *B*

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Investment Opportunity Set

	A	B	C	D	E
1			**Input Data**		
2	$E(r_S)$	$E(r_B)$	σ_S	σ_B	ρ_{BS}
3	10	5	19	8	0.2
4	Portfolio Weights		Expected Return, $E(r_p)$		Std Dev
5	$w_S = 1 - w_B$	w_B	Col A*A3 + Col B*B3		(Equation 6.6)
6	−0.2	1.2	4.0		9.59
7	−0.1	1.1	4.5		8.62
8	0.0	1.0	5.0		8.00
9	0.0932	0.9068	5.5		7.804
10	0.1	0.9	5.5		7.81
11	0.2	0.8	6.0		8.07
12	0.3	0.7	6.5		8.75
13	0.4	0.6	7.0		9.77
14	0.5	0.5	7.5		11.02
15	0.6	0.4	8.0		12.44
16	0.7	0.3	8.5		13.98
17	0.8	0.2	9.0		15.60
18	0.9	0.1	9.5		17.28
19	1.0	0.0	10.0		19.00
20	1.1	−0.1	10.5		20.75
21	1.2	−0.2	11.0		22.53
22	Notes:				
23	1. Negative weights indicate short positions.				
24	2. The weights of the minimum-variance portfolio are computed using the formula in Footnote 1.				

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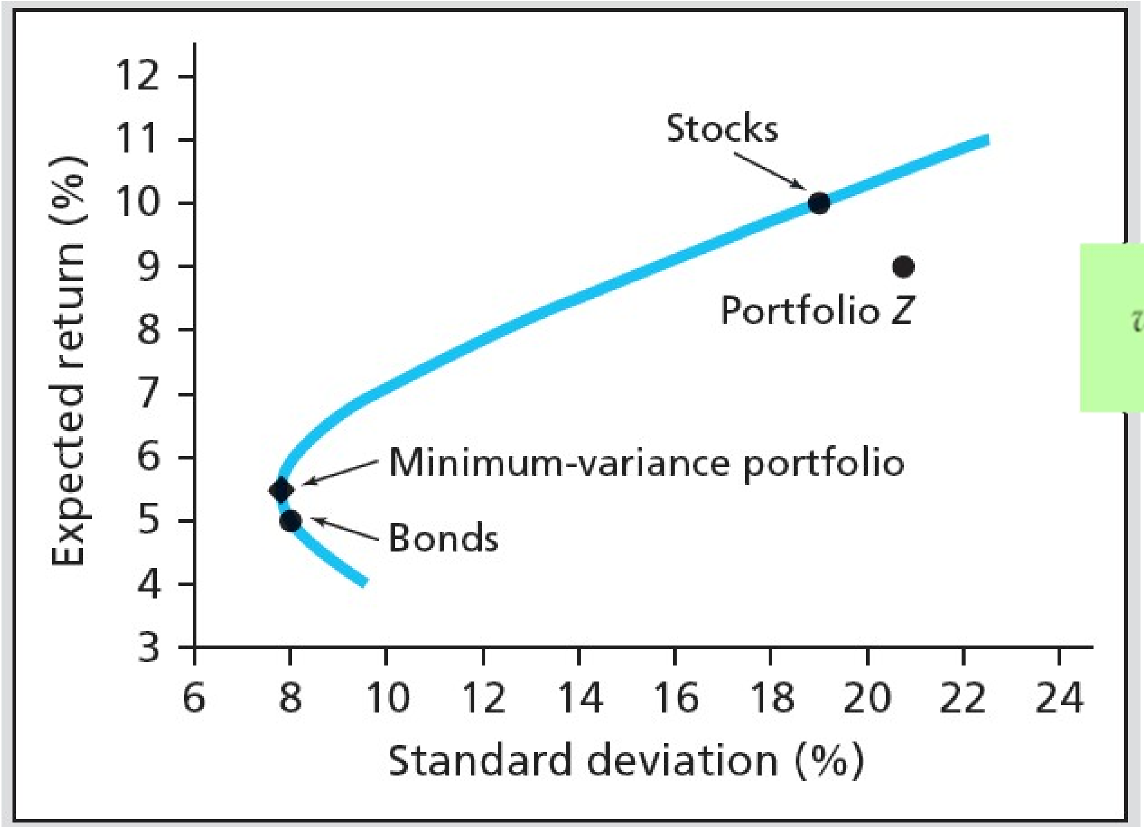
Efficient Portfolio

- A **portfolio is efficient** if (and only if) there is **no alternative** with
  - (i) the same *E*(R_p) and a lower σ_p,
  - (ii) the same σ_p and a higher *E*(R_p) , or
  - (iii) a higher *E*(R_p) and a lower *E*(R_p) .

Investment Opportunity Set



$$w_b(\min) = \frac{\sigma_s^2 - \sigma_b\sigma_s\rho_{bs}}{\sigma_b^2 + \sigma_s^2 - 2\sigma_b\sigma_s\rho_{bs}}$$

Opportunity Set -Various Correlation Coefficients

	A	B	C	D	E	F	G
1		Input Data					
2	$E(r_S)$	$E(r_B)$	σ_S	σ_B			
3	10	5	19	8			
4							
5	Weights in Stocks	Portfolio Expected Return	Portfolio Standard Deviation[1] for Given Correlation, ρ				
6	w_S	$E(r_P)$ = Col A*A3 + (1 − Col A)*B3	−1	0	0.2	0.5	1
7	−0.1	4.5	10.70	9.00	8.62	8.02	6.90
8	0.0	5.0	8.00	8.00	8.00	8.00	8.00
9	0.1	5.5	5.30	7.45	7.81	8.31	9.10
10	0.2	6.0	2.60	7.44	8.07	8.93	10.20
11	0.3	6.5	0.10	7.99	8.75	9.79	11.30
12	0.4	7.0	2.80	8.99	9.77	10.83	12.40
13	0.6	8.0	8.20	11.84	12.44	13.29	14.60
14	0.8	9.0	13.60	15.28	15.60	16.06	16.80
15	1.0	10.0	19.00	19.00	19.00	19.00	19.00
16	1.1	10.5	21.70	20.92	20.75	20.51	20.10
17				Minimum-Variance Portfolio[2,3,4,5]			
18	$w_S(\min) = (\sigma_B{}^\wedge 2 - \sigma_B\sigma_S\rho)/(\sigma_S{}^\wedge 2 + \sigma_B{}^\wedge 2 - 2^*\sigma_B\sigma_S\rho) =$		0.2963	0.1506	0.0923	−0.0440	−0.7273
19	$E(r_P) = w_S(\min)^*A3 + (1 - w_S(\min))^*B3 =$		6.48	5.75	5.46	4.78	1.36
20		$\sigma_P =$	0.00	7.37	7.80	7.97	0.00

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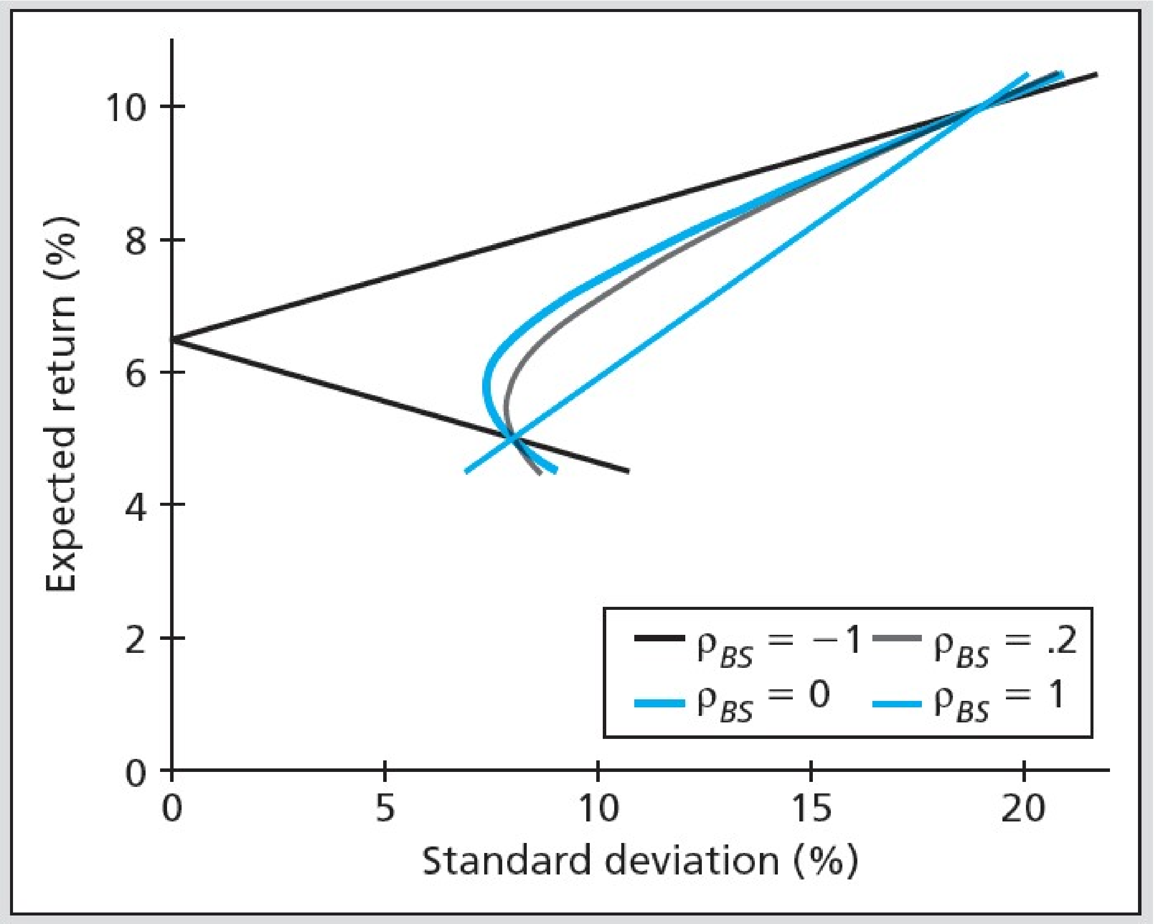
Opportunity Sets: Various Correlation Coefficients



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Standard Deviation of Various Correlations

In case 1, $\rho_{ab} = 1$.

So

$$\sigma_p = [w_a^2\,\sigma_a^2 + w_b^2\,\sigma_b^2 + 2\,w_a\,w_b\,\sigma_a\,\sigma_b]^{1/2}$$

$$= [w_a\,\sigma_a + w_b\,\sigma_b]$$

In case 2, $\rho_{ab} < 1$.

So

$$\sigma_p = [w_a^2\,\sigma_a^2 + w_b^2\,\sigma_b^2 + 2\,w_a\,w_b\,\rho_{ab}\,\sigma_a\,\sigma_b]^{1/2}$$

Since $\rho_{ab} < 1$, σ_p will be less than $[w_a\,\sigma_a + w_b\,\sigma_b]$

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Efficient Frontier for the n-Security Case

- In a **2-security case**, a curved line delineates all possible portfolios.
- In a **multi-security case**, the collection of all the possible portfolios is represented by the a cloud region, referred to as the feasible region.

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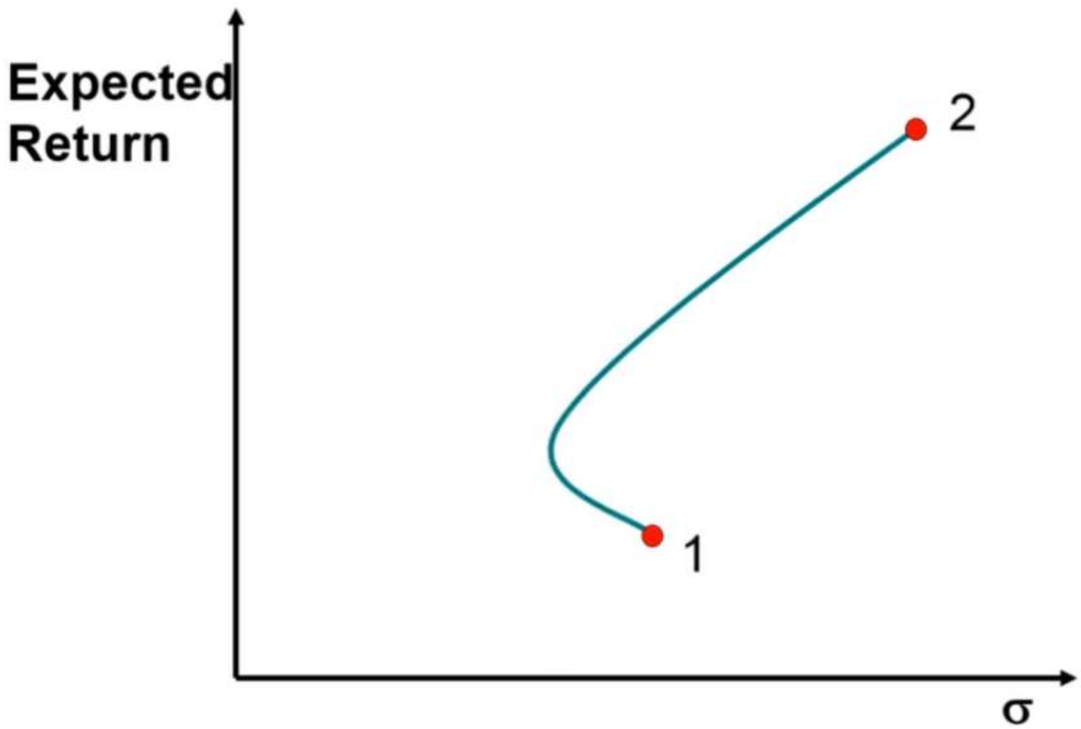
Efficient Frontier for the n-Security Case



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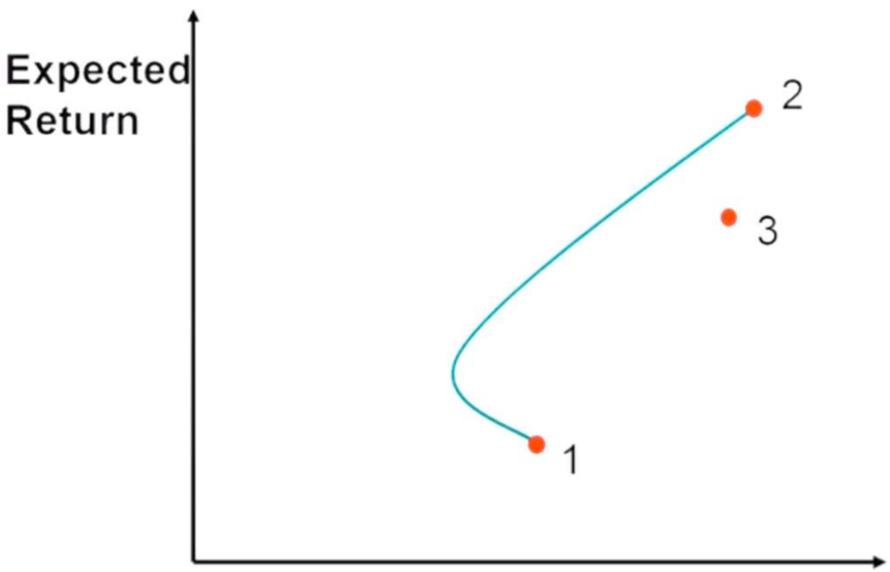
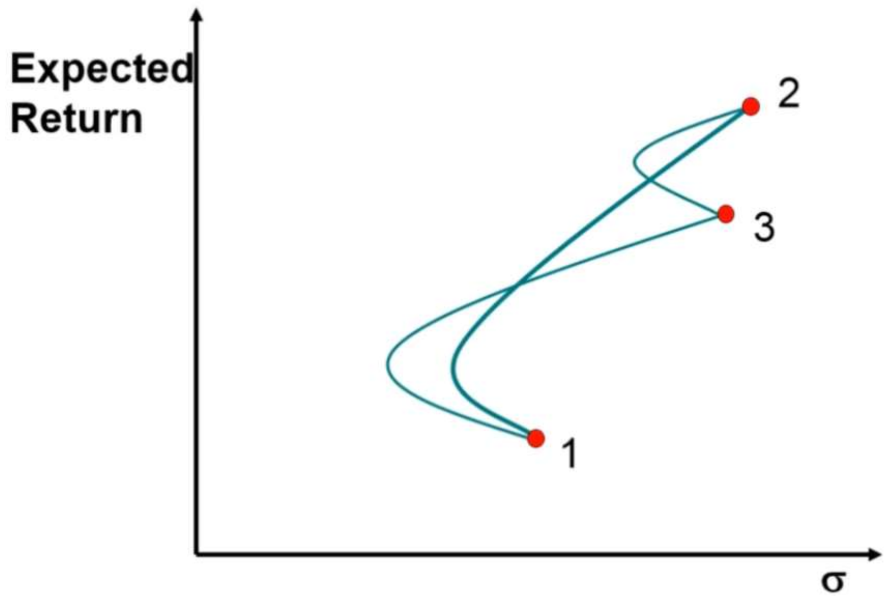
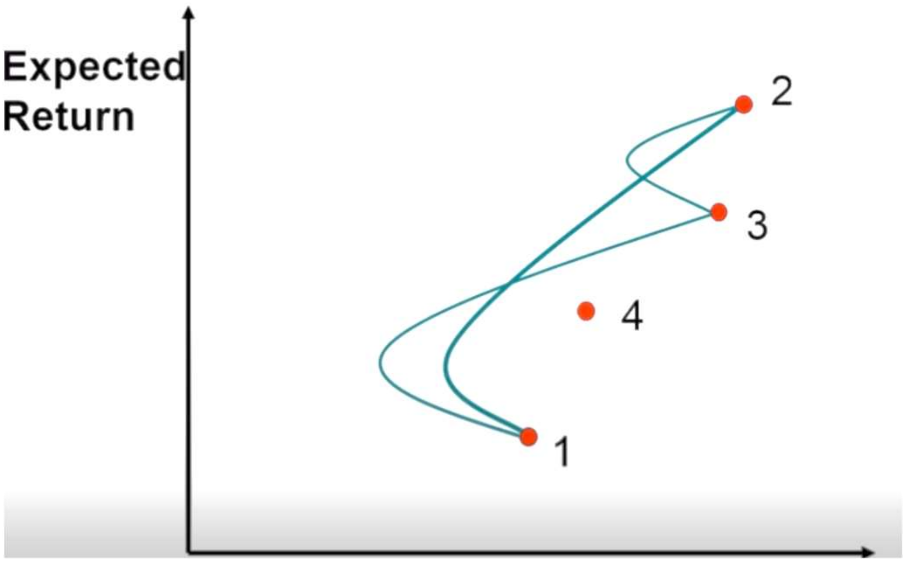
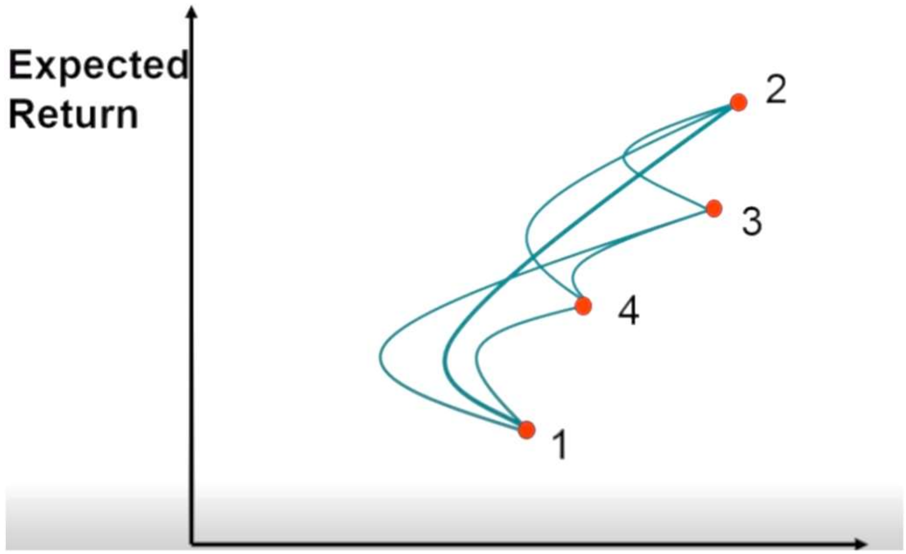
Efficient Frontier for the n-Security Case





Efficient Frontier for the n-Security Case





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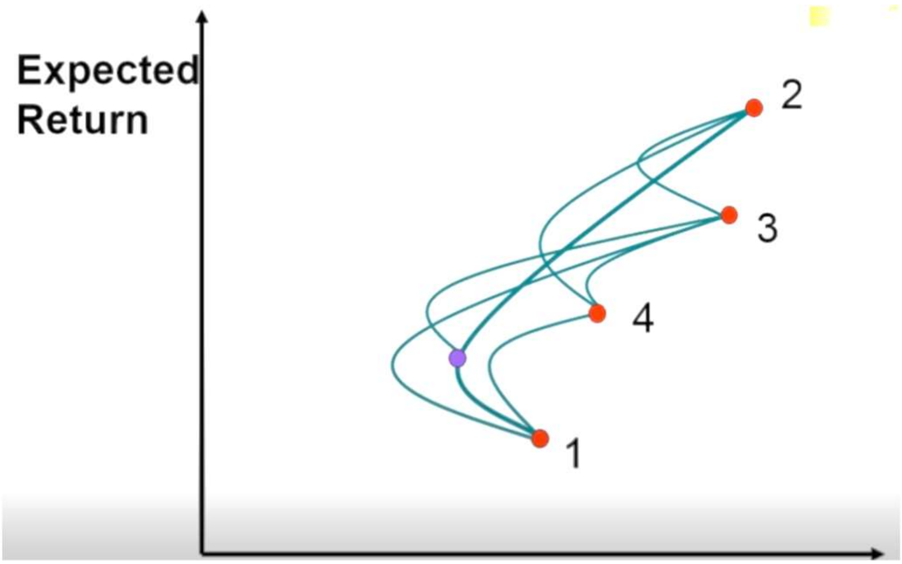
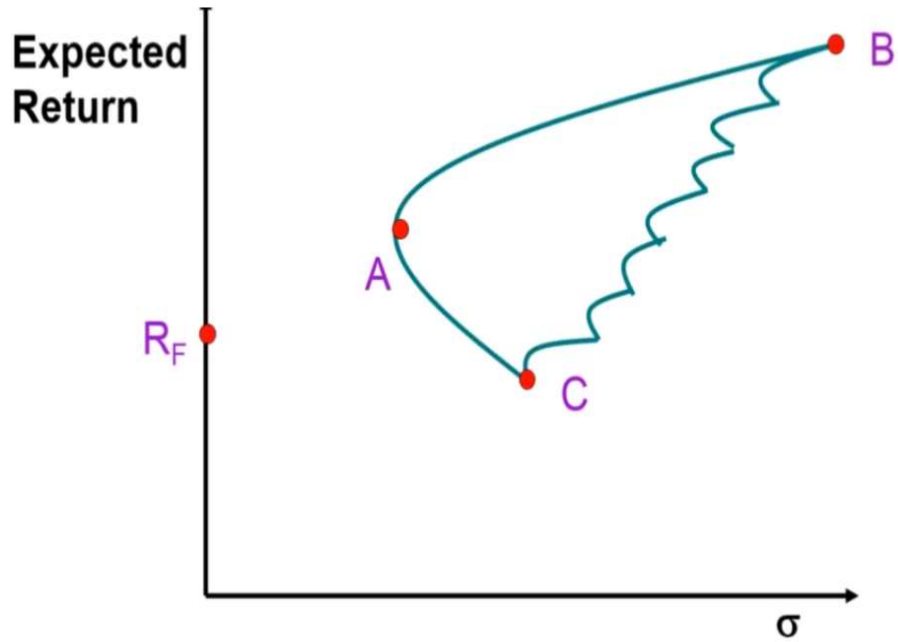
Efficient Frontier for the n-Security Case





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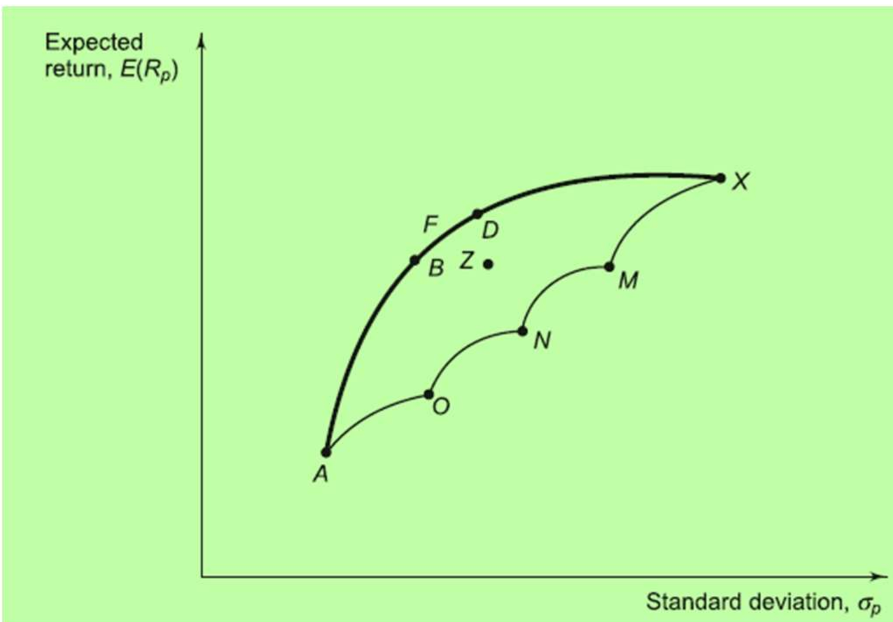
Efficient Portfolios



- **AFX are efficient**. AFX represents the efficient frontier.
  - All other portfolios are inefficient.
- A portfolio like **Z is inefficient** because portfolios like B and D, among others, dominate it.
- The efficient frontier is the same for all investors because **portfolio theory** is based on the **assumption** that investors have **homogeneous expectations**.

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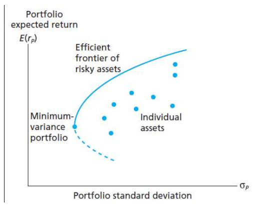
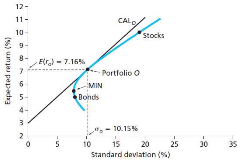
Summary of Portfolio Construction Process

Process

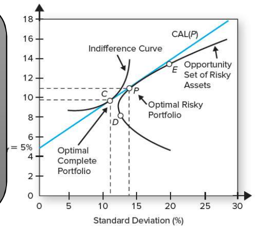
1. Determine the Efficient Frontier of Risky Assets



2. Determine the Optimal Risky Portfolio (Pº)



Determine the optimal Allocation between the ORP and the Risk Free Asset



$$E(r_P) = w_B E(r_B) + w_S E(r_S)$$

$$\sigma_P^2 = (w_B\sigma_B)^2 + (w_S\sigma_S)^2 + 2(w_B\sigma_B)(w_S\sigma_S)\rho_{BS}$$

$$w_B = \frac{\sigma_S^2 - \sigma_B\sigma_S\rho_{BS}}{\sigma_S^2 + \sigma_B^2 - 2\sigma_B\sigma_S\rho_{BS}}$$

$$w_S = 1 - w_B$$

Weights of the GMVP

$$w_B = \frac{Var(r_S) - Cov(r_S, r_B)}{Var(r_B) + Var(r_S) - 2Cov(r_S, r_B)}$$

$$S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

$$w_B = \frac{[E(r_B) - r_f]\sigma_S^2 - [E(r_s) - r_f]\sigma_B\sigma_S\rho_{BS}}{[E(r_B) - r_f]\sigma_S^2 + [E(r_s) - r_f]\sigma_B^2 - [E(r_B) - r_f + E(r_s) - r_f]\sigma_B\sigma_S\rho_{BS}}$$

$$w_S = 1 - w_B$$

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

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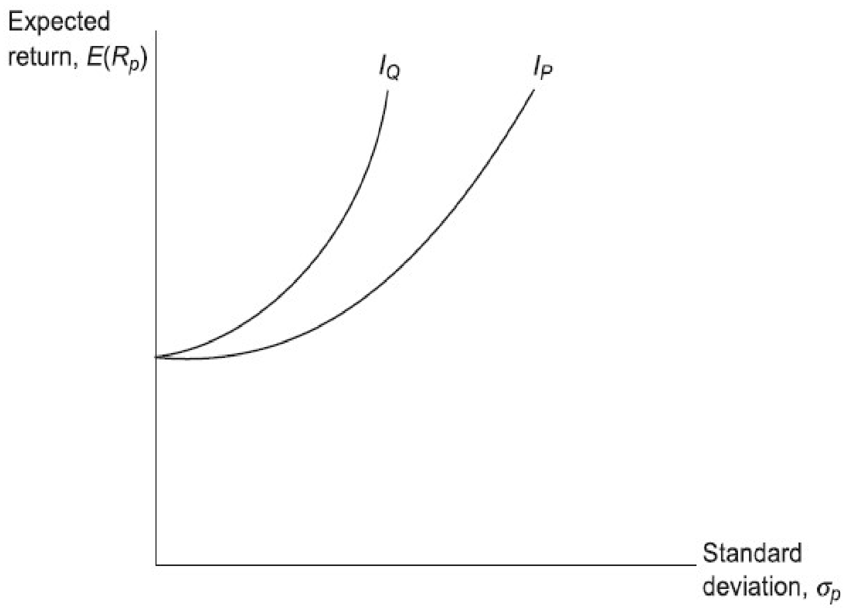
Optimal Portfolio

- Once the efficient frontier is delineated, the next question is: What is the optimal portfolio for the investor?

- To determine the optimal portfolio on the efficient frontier, the **investor's risk-return tradeoff** must be known.

Optimal Portfolio

- **Indifference curves** which reflect risk- return tradeoff functions.
    - All points lying on an indifference curve **provide the same level of satisfaction.**



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Optimal Portfolio

- Q wants a higher expected return for bearing a given amount of risk as compared to P.
- In general, **the steeper the slope** of the indifference curve, the **greater the degree of risk aversion**.

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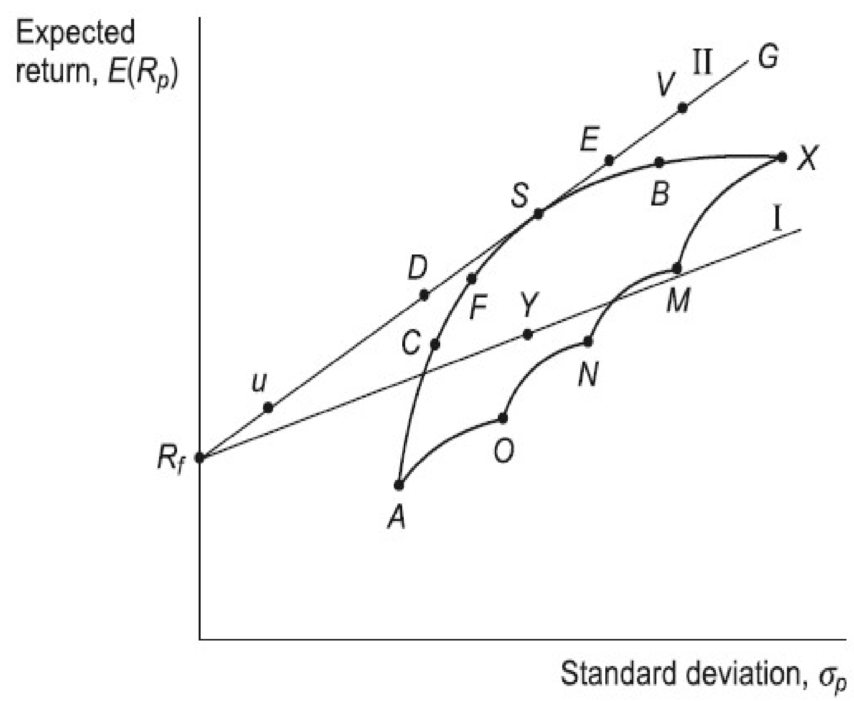
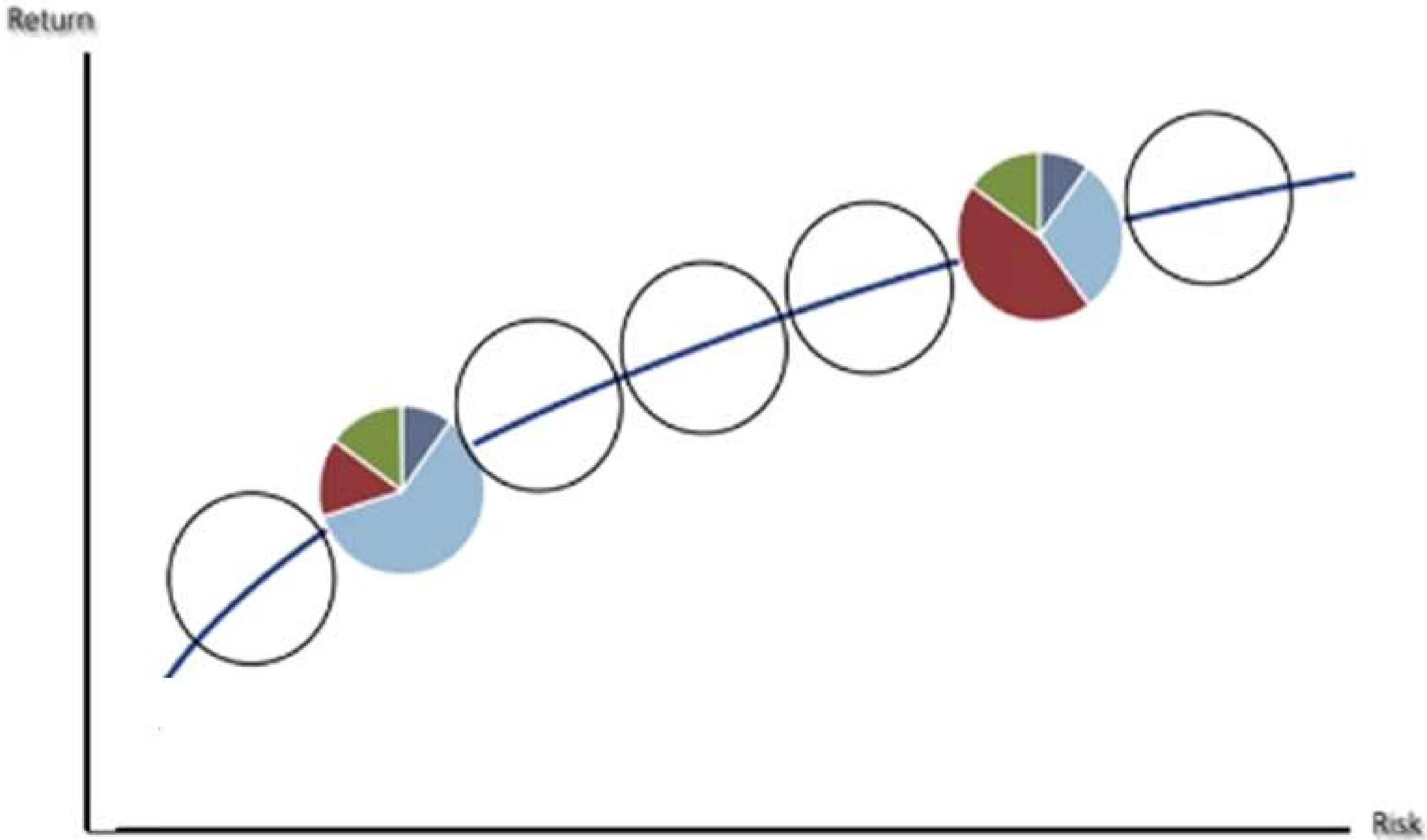
RISKLESS LENDING AND BORROWING

- Let us introduce yet another opportunity. Suppose that investors can also **lend and borrow money at a risk-free rate of** R_f.
- Since Rf is a risk-free asset it has **a zero correlation** with all the points in the feasible region of risky portfolios.
  - So a combination of Rf and any point in the feasible region of risky securities will be **represented by a straight line**.




Return
Risk

RISKLESS LENDING AND BORROWING

- With the **opportunity of lending and borrowing**, the efficient frontier changes.
- It is no longer AFX. Rather, it is Rf SG because Rf SG dominates AFX.
- For every point on AFX (excepting S) there is at least one point on Rf SG which is superior to the point on AFX.
- Since R_fSG **dominates** AFX, **every investor would do well to choose some combination of R_f and S**

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Optimal Risky Portfolio

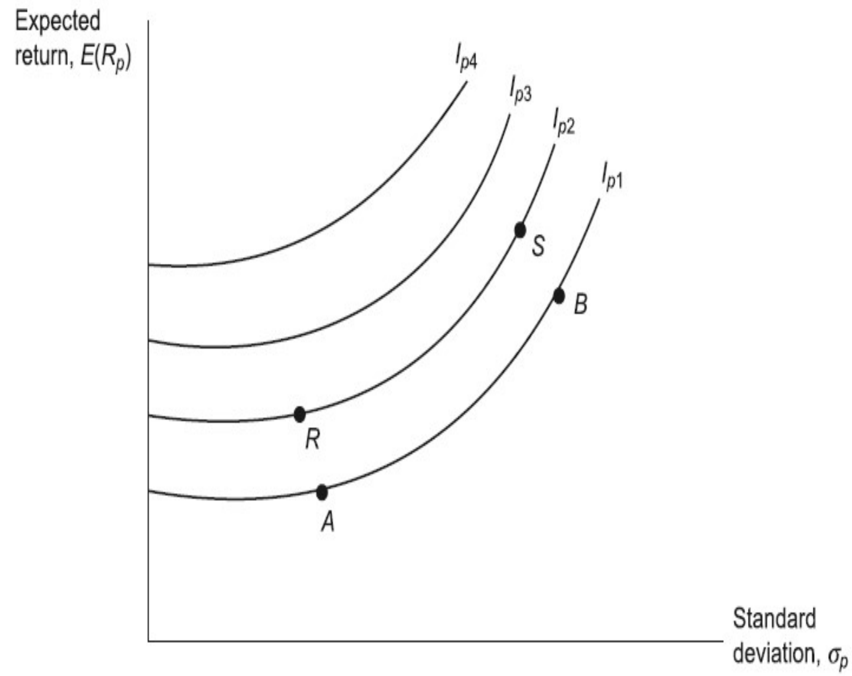
$$S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

$$w_B = \frac{[E(r_B)-r_f]\sigma_S^2 - [E(r_s)-r_f]\sigma_B\sigma_S\rho_{BS}}{[E(r_B)-r_f]\sigma_S^2 + [E(r_s)-r_f]\sigma_B^2 - [E(r_B)-r_f + E(r_s)-r_f]\sigma_B\sigma_S\rho_{BS}}$$

$$w_S = 1 - w_B$$

Optimal Complete Portfolio

- Each person has a map of indifference curves.
- All the points lying on a given indifference curve **offer the same level of satisfaction.**
- The level of **satisfaction increases** as one moves leftward.



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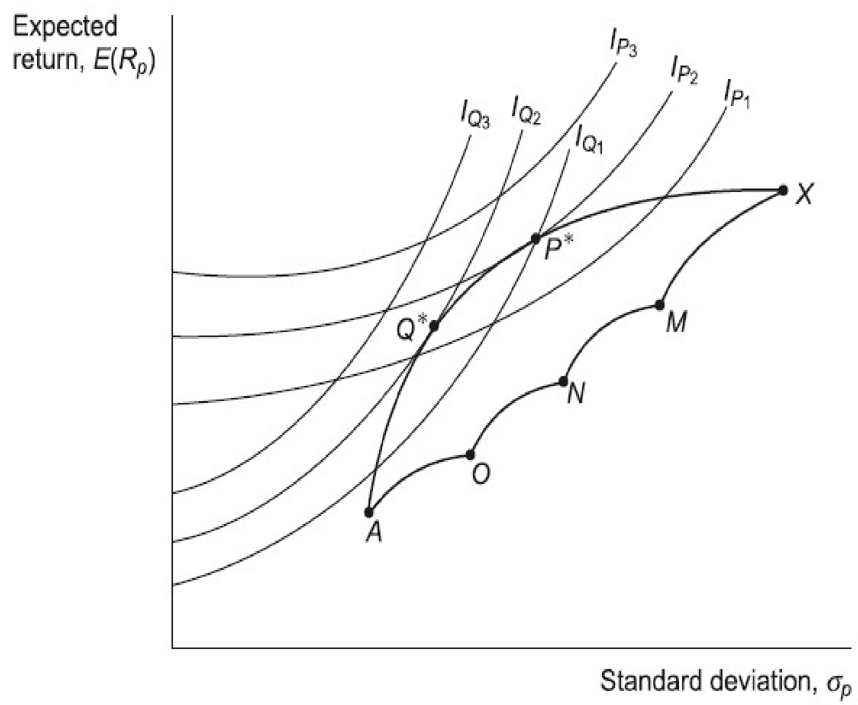
Optimal Complete Portfolio

- Given the efficient frontier and the risk-return indifference curves, the **optimal portfolio** is found at **the point of tangency** between the **efficient frontier and a utility indifference curve**.



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Optimal Complete Portfolio

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

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Capital Allocation and the Separation Property

- The separation property tells us that the portfolio choice problem may be separated into two independent tasks:

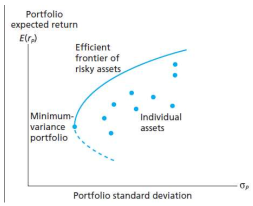
  1. **Determination of <u>the optimal risky portfolio (P^o)</u> is purely technical.**

  2. **<u>Allocation of the complete portfolio</u>** to T-bills versus the risky portfolio depends on personal preference.

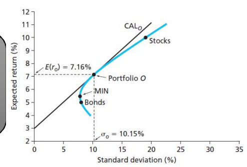
Summary of Portfolio Construction Process

Process

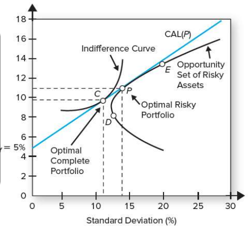
1. Determine the Efficient Frontier of Risky Assets



2. Determine the Optimal Risky Portfolio (P°)



Determine the optimal Allocation between the ORP and the Risk Free Asset



$$E(r_P) = w_B E(r_B) + w_S E(r_S)$$

$$\sigma_P^2 = (w_B\sigma_B)^2 + (w_S\sigma_S)^2 + 2(w_B\sigma_B)(w_S\sigma_S)\rho_{BS}$$

$$w_B = \frac{\sigma_S^2 - \sigma_B\sigma_S\rho_{BS}}{\sigma_S^2 + \sigma_B^2 - 2\sigma_B\sigma_S\rho_{BS}}$$

$$w_S = 1 - w_B$$

Weights of the GMVP

$$w_B = \frac{Var(r_S) - Cov(r_S, r_B)}{Var(r_B) + Var(r_S) - 2Cov(r_S, r_B)}$$

$$S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

$$w_B = \frac{[E(r_B) - r_f]\sigma_S^2 - [E(r_s) - r_f]\sigma_B\sigma_S\rho_{BS}}{[E(r_B) - r_f]\sigma_S^2 + [E(r_s) - r_f]\sigma_B^2 - [E(r_B) - r_f + E(r_s) - r_f]\sigma_B\sigma_S\rho_{BS}}$$

$$w_S = 1 - w_B$$

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

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Example 3: p.164

CONCEPT check 6.4

A universe of securities includes a risky stock (*X*), a stock-index fund (*M*), and T-bills. The data for the universe are:

	Expected Return	Standard Deviation
X	15%	50%
M	10	20
T-bills	5	0

The correlation coefficient between *X* and *M* is −.2.

a. Draw the opportunity set of securities *X* and *M*.

b. Find the optimal risky portfolio (*O*), its expected return, standard deviation, and Sharpe ratio. Compare with the Sharpe ratio of *X* and *M*.

c. Find the slope of the CAL generated by T-bills and portfolio *O*.

d. Suppose an investor places 2/9 (i.e., 22.22%) of the complete portfolio in the risky portfolio *O* and the remainder in T-bills. Calculate the composition of the complete portfolio, its expected return, SD, and Sharpe ratio.

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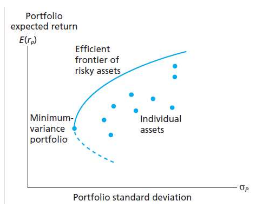
Step1: Determine the MVF

Process

1. Determine the Efficient Frontier of Risky Assets



$$E(r_P) = w_B E(r_B) + w_S E(r_S)$$

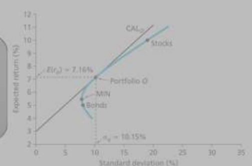
$$\sigma_P^2 = (w_B\sigma_B)^2 + (w_S\sigma_S)^2 + 2(w_B\sigma_B)(w_S\sigma_S)\rho_{BS}$$

$$w_B = \frac{\sigma_S^2 - \sigma_B\sigma_S\rho_{BS}}{\sigma_S^2 + \sigma_B^2 - 2\sigma_B\sigma_S\rho_{BS}}$$

$$w_S = 1 - w_B$$

Weights of the GMVP

2. Determine the Optimal Risky Portfolio (P^o)

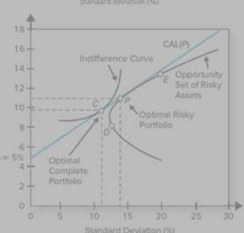


$$S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

$$w_B = \frac{[E(r_B)-r_f]\sigma_S^2 - [E(r_S)-r_f]\sigma_B\sigma_S\rho_{BS}}{[E(r_B)-r_f]\sigma_S^2 + [E(r_S)-r_f]\sigma_B^2 - [E(r_B)-r_f+E(r_S)-r_f]\sigma_B\sigma_S\rho_{BS}}$$

$$w_S = 1 - w_B$$

Determine the optimal Allocation between the ORP and the Risk Free Asset



$$y = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

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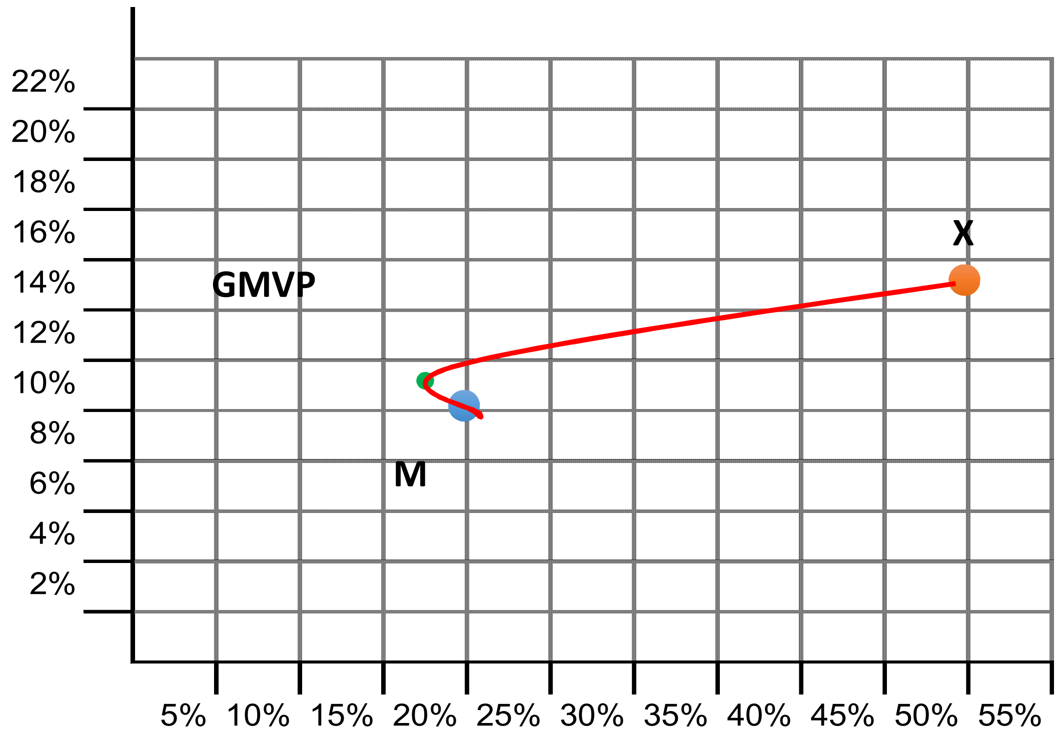


$$w_M = \frac{Var(r_X) - Cov(r_X, r_M)}{Var(r_X) + Var(r_M) - 2\,Cov(r_X, r_M)}$$

First: We find the Cov (X,M)		
Cov	-0.02	

$$W_M = \frac{0.5^2 - (-0.02)}{0.5^2 + 0.2^2 - 2 \times (-0.02)} = \mathbf{81.8}\%$$

$$W_X = 1 - 0.818 = 18.2\%$$

$$E(R_{GMVP}) = 0.15 \times 0.182 + 0.10 \times .818 = 10.9\%$$

$$\sigma_{GMVP} = \sqrt{\sigma^2_{GMVP}}$$

$$\sigma^2_{GMVP} = 0.182^2 \times 0.5^2 + 0.818^2 \times 0.20^2 + 2 \times 0.182 \times 0.818 \times (-0.02) = 0.0291$$

$$\sigma_{GMVP} = 17.1\%$$

Step2: Determine the ORP

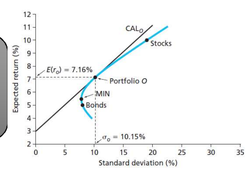
Process

1. Determine the Efficient Frontier of Risky Assets

$$E(r_P) = w_B E(r_B) + w_S E(r_S)$$
$$\sigma_P^2 = (w_B \sigma_B)^2 + (w_S \sigma_S)^2 + 2(w_B \sigma_B)(w_S \sigma_S)\rho_{BS}$$
$$w_B = \frac{\sigma_S^2 - \sigma_B \sigma_S \rho_{BS}}{\sigma_S^2 + \sigma_B^2 - 2\sigma_B \sigma_S \rho_{BS}}$$
$$w_S = 1 - w_B$$

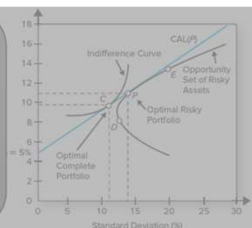
Weights of the GMVP

2. Determine the Optimal Risky Portfolio (P^o)



$$S_P = \frac{E(r_P) - r_f}{\sigma_P}$$
$$w_B = \frac{[E(r_B) - r_f]\sigma_S^2 - [E(r_s) - r_f]\sigma_B \sigma_S \rho_{BS}}{[E(r_B) - r_f]\sigma_S^2 + [E(r_s) - r_f]\sigma_B^2 - [E(r_B) - r_f + E(r_s) - r_f]\sigma_B \sigma_S \rho_{BS}}$$
$$w_S = 1 - w_B$$

Determine the optimal Allocation between the ORP and the Risk Free Asset



$$y = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

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$$w_M = \frac{[E(r_M) - r_f]\sigma_X^2 - [E(r_X) - r_f]\sigma_M \sigma_X \rho_{MX}}{[E(r_M) - r_f]\sigma_X^2 + [E(r_X) - r_f]\sigma_M^2 - [E(r_M) - r_f + E(r_X) - r_f]\sigma_M \sigma_X \rho_{MX}}$$

$$w_X = 1 - w_M$$

$$W_M = \frac{[0.10 - 0.05] \times 0.50^2 - [0.15 - .05] \times (-0.02)}{[0.10 - 0.05] \times 0.50^2 + [0.15 - 0.05] \times 0.20^2 - [(0.10 - .05) + (0.15 - .05) \times (-0.02)]}$$

$$W_M = \frac{0.0145}{0.0195} = 74.36\%$$

$$W_X = 25.64\%$$

$$E(R_{P^o}) = 0.15 \times 0.2564 + 0.10 \times 0.7436 = 11.3\%$$

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$$\sigma_{P^o} = \sqrt{\sigma_{P^o}^2}$$

$$\sigma_{P^o}^2 = 0.2564^2 \times 0.5^2 + 0.7436^2 \times 0.20^2 + 2 \times 0.2564 \times 0.7436 \times (-0.02) = 0.0309$$

$$\sigma_{P^o} = 17.59\%$$

$$SR_{P^o} = \frac{E(R_{P^o}) - R_f}{\sigma_{P^o}} = \frac{0.113 - .05}{0.1759} = 0.3572$$

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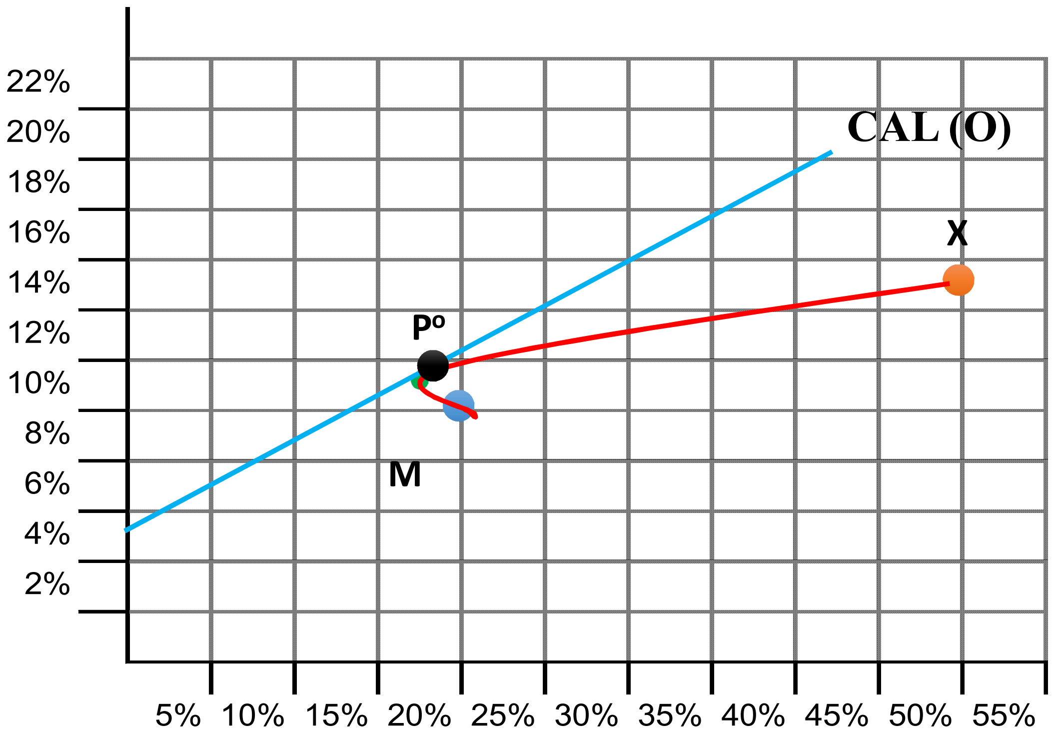

$$E(R_{P^o}) = 11.3\%$$

$$\sigma_{P^o} = 17.59\%$$

$$SR_{P^o} = 0.3572$$

Example 3: p.164

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c. Find the slope of the CAL generated by T-bills and portfolio *O*.

d. Suppose an investor places 2/9 (i.e., 22.22%) of the complete portfolio in the risky portfolio *O* and the remainder in T-bills. Calculate the composition of the complete portfolio, its expected return, SD, and Sharpe ratio.

$$E(R_{P^o}) = 11.3\%$$

$$\sigma_{P^o} = 17.59\%$$

$$SR_{P^o}=0.3572$$

$$E(R_c) = R_f + \left[\frac{E(R_{P^o}) - R_f}{\sigma_{P^o}}\right] \times \sigma_c$$

$$\sigma_c = Y\sigma_{P^o}$$

Where Y= % invested in the optimal risky portfolio (22.22%)

1) $\sigma_c = 0.2222 \times 0.1759 = 3.91\%$

$$E(R_c) = 0.05 + \left[\frac{0.113-.05}{.1759}\right] \times .0391=6.40\%$$

The composition of the complete portfolio is

0.2222*0.2564 =5.70% in X

0.2222*0.7436 =16.52% in M

77.77% in the Risk Free Asset

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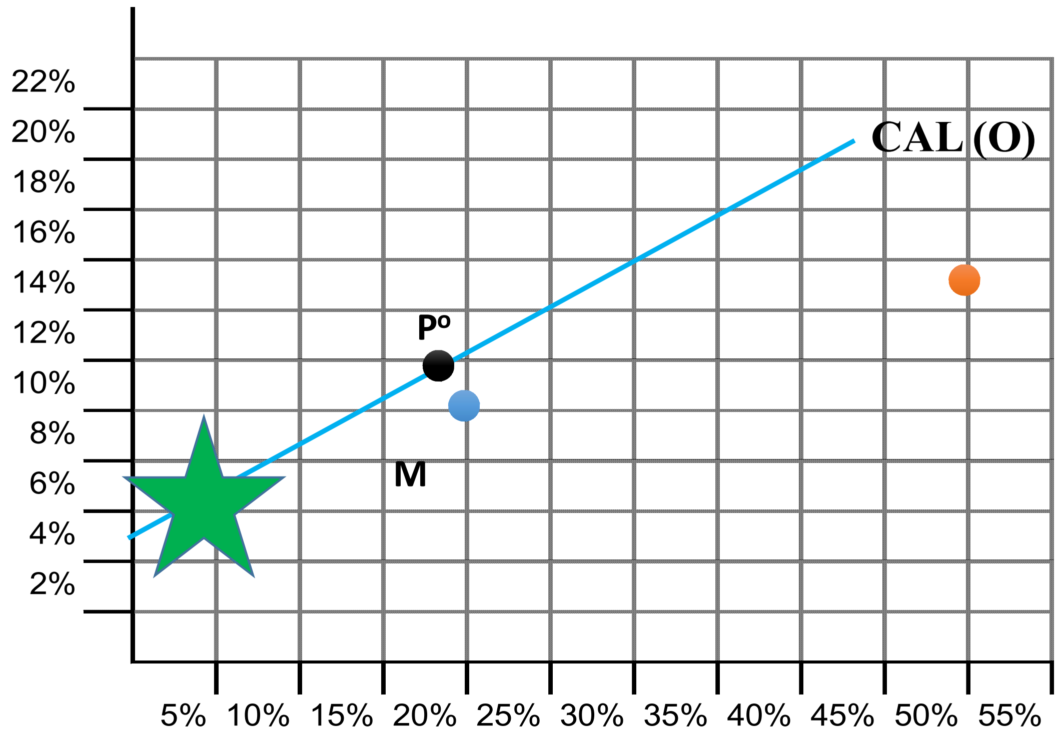
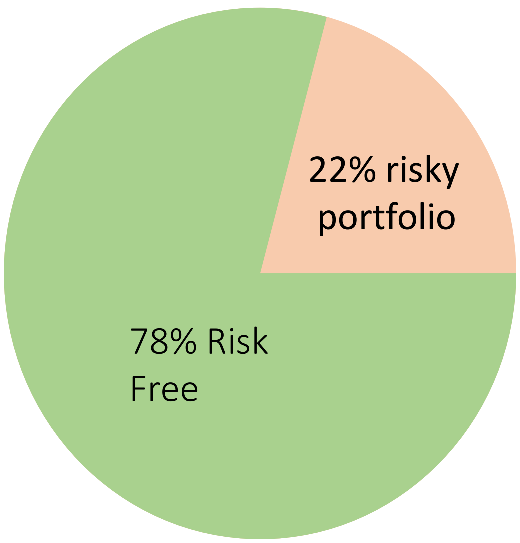
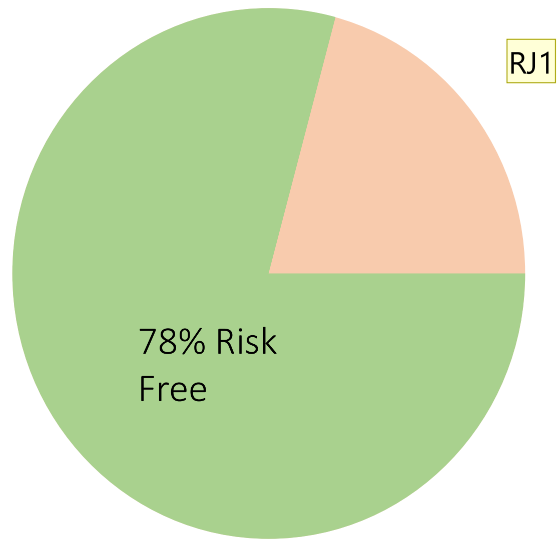


$$E(R_c) = 6.40\%$$

$$\sigma_c = 3.91\%$$

$$SR_c = 0.3572$$


22% risky portfolio
78% Risk Free


RJ1
78% Risk Free

Slide 72

RJ1 Dr. Ra'fat Jallad, 7/11/2020

Single Index Model
(SIM)

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1. With n risky assets, we need $2n + (n^2 - n)/2$ parameters:

n	expected returns $E[r_i]$
n	return standard deviations σ_i
$n(n-1)/2$	correlations (or covariances)

Example

$n = 2$	number of parameters = 2 + 2 + 1 = 5
$n = 8$	number of parameters = 8 + 8 + 28 = 44
$n = 100$	number of parameters = 100 + 100 + 4950 = 5150
$n = 1000$	number of parameters = 1000+ 1000 + 499500= 501500

With large n:

Large estimation error,

Large data requirements (for monthly estimates, with n=1000, need at least 1000 months, i.e., more than 83 years of data)

SIM: Model's Components

- **1. The Basic Idea:** Stocks tend to move together, driven by the same economic forces.

$$\boxed{R_i = \alpha_i + \beta_i R_M + e_i}$$

$E[e_i] = 0$, $Cov[e_{i}, R_m] = 0$

R_i : stock's Return

α_i : stock's expected excess return if market factor is neutral, i.e. market-index excess return is zero

β_i : security's responsiveness to market $\beta_i = Cov[R_i, R_m]/Var[R_m]$

$\beta_i R_M$: return from movements in overall market

e_i : firm-specific risk

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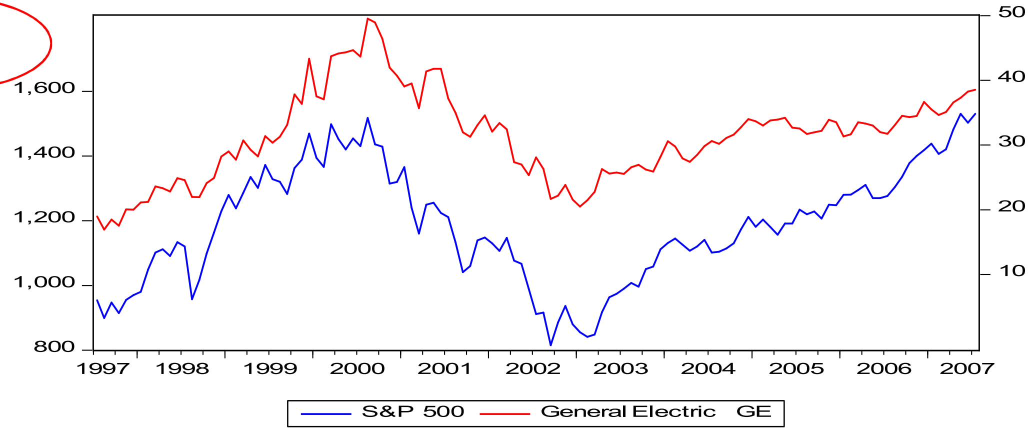
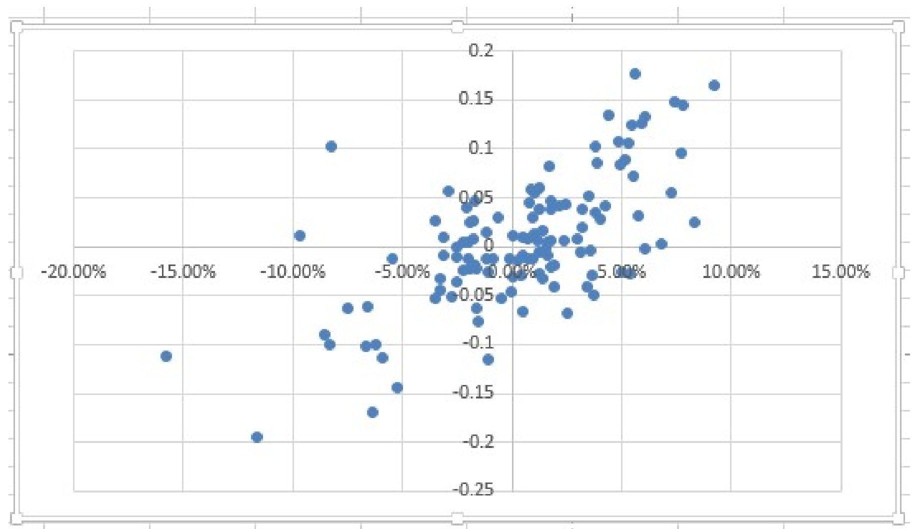
R_m R_i

	Date	S&P 500	General Electric GE
1	Date	S&P 500	General Electric GE
2	1-Aug-97	-5.92%	-0.1137716
3	2-Sep-97	5.18%	0.0881428
4	1-Oct-97	-3.51%	-0.0521761
5	3-Nov-97	4.36%	0.1336738
6	1-Dec-97	1.56%	-0.0024956
7	2-Jan-98	1.01%	0.0549348
8	2-Feb-98	6.81%	0.0033058
9	2-Mar-98	4.87%	0.1067739
10	1-Apr-98	0.90%	-0.0115066
11	1-May-98	-1.90%	-0.0216646
12	1-Jun-98	3.87%	0.0859888
13	1-Jul-98	-1.17%	-0.0125381
14	3-Aug-98	-15.76%	-0.1113932
15	1-Sep-98	6.05%	-0.0018215
16	1-Oct-98	7.72%	0.0951444
17	2-Nov-98	5.74%	0.0322148
18	1-Dec-98	5.48%	0.1243603
19	4-Jan-99	4.02%	0.0279543
20	1-Feb-99	-3.28%	-0.0447494
21	1-Mar-99	3.81%	0.1016781
22	1-Apr-99	3.72%	-0.0486925
23	3-May-99	-2.53%	-0.035482
24	1-Jun-99	5.30%	0.1054313
25	1-Jul-99	-3.26%	-0.0330448
26	2-Aug-99	-0.63%	0.029853
27	1-Sep-99	-2.90%	0.0571584
28	1-Oct-99	6.07%	0.1334937





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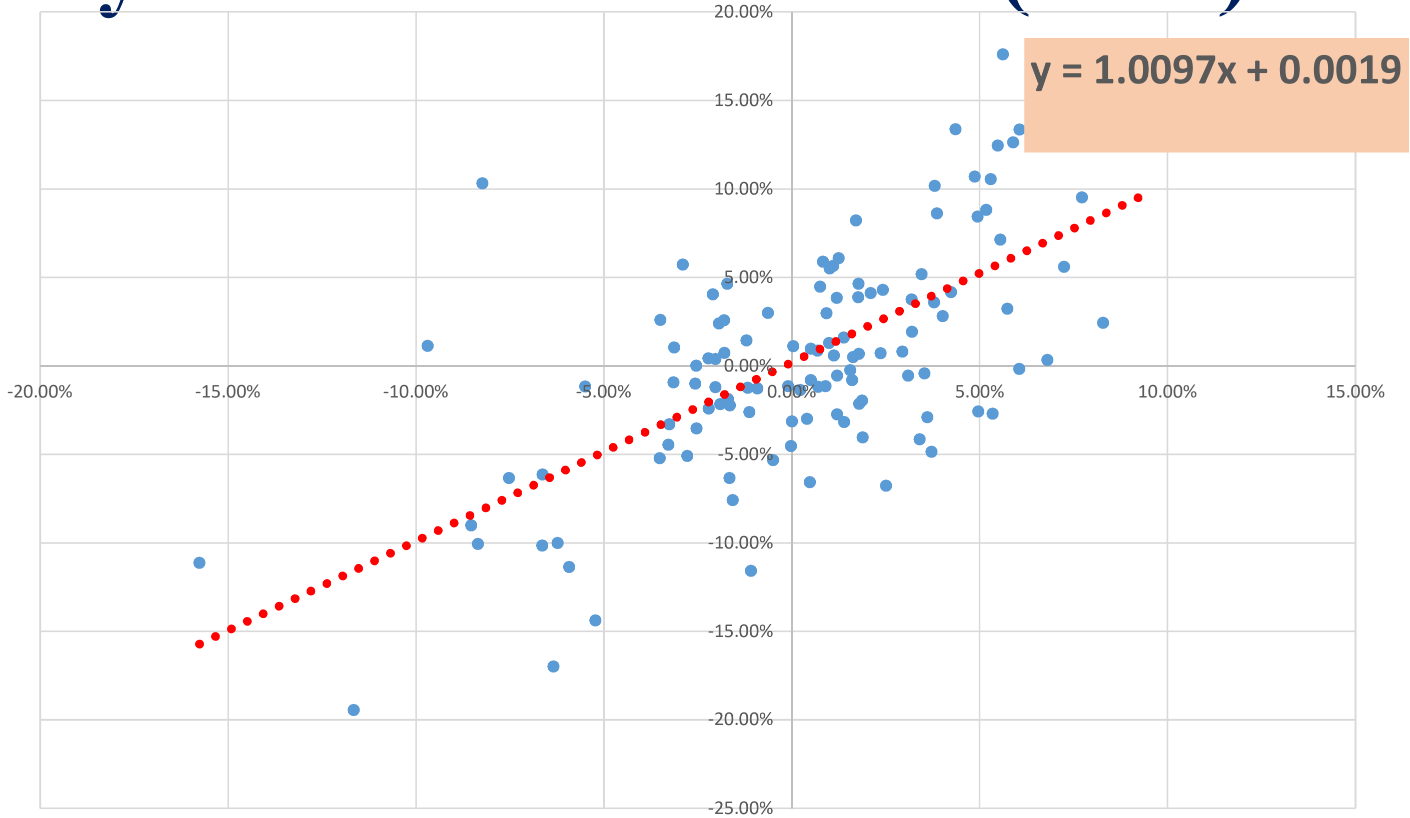
Security Characteristic Line (SCL)



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Single Index Model (SIM)

- An Index Model is a statistical model of security returns *(as opposed to an economic, equilibrium-based model).*
- A Single Index Model (SIM) specifies two sources of uncertainty for a security's return:
  - 1. **Systematic (macroeconomic) uncertainty** (which is assumed to be well represented by a single index of stock returns).
  - 2. **Unique (microeconomic) uncertainty** (which is represented by a security-specific random component)

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Expressing the expected return, variance and covariance

1. Mean return of security *i*: $E[r_i] = \alpha_i + \beta_i E[R_m]$

2. Variance of security *i*: $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma^2[e_i]$

3. Covariance between return of security *j* and return of security *i*
$$\sigma_{ji} = \beta_j \beta_i \sigma_m^2$$

Cont'd

- **2. Assuming the SIM is correctly specified, we only need the following parameters:**

n	α_j parameters
n	β_j parameters
n	$\sigma^2[e_j]$ parameters
1	$E[Rm]$
1	$\sigma^2[Rm]$

These $3n+2$ parameters generate all the $E[r_j]$, σ_j, and σ_{ji}.
We get the parameters by estimating the index model for each of the n securities.

Example

With 100 stocks need 302 parameters. With 1000 need 3002.

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Expressing the expected return, variance and covariance

1. Mean return of security i: $E[r_i] = \alpha_i + \beta_i E[R_m]$

2. Variance of security i: $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma^2[e_i]$

3. Covariance between return of security j and return of security i

$$\sigma_{ji} = \beta_j \beta_i \sigma_m^2$$

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Formalizing the Basic Idea

1. ***Expected Return,*** $E[r_j] = \alpha_j + \beta_j E[R_m]$, ***has 2 parts***

   a. Unique (asset specific): α_j

   b. Systematic (index driven): $\beta_j E[R_m]$

2. ***Variance,*** $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma^2[e_i]$, ***has similarly 2 parts:***

   a. Unique risk (asset specific): $\sigma^2[e_i]$

   b. Systematic risk (index driven): $\beta_i^2 \sigma_m^2$

3. ***Covariance between securities' returns is due to only the systematic source of risk:***

$$\mathrm{Cov}[r_j, r_i] = \beta_j \beta_i \sigma_m^2$$

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Example-GE

The fitted regression line is called the ***Security Characteristic Line*** (SCL).

GE Equity BETA
Screen printed.

DG21 Equity **BETA**

HISTORICAL BETA

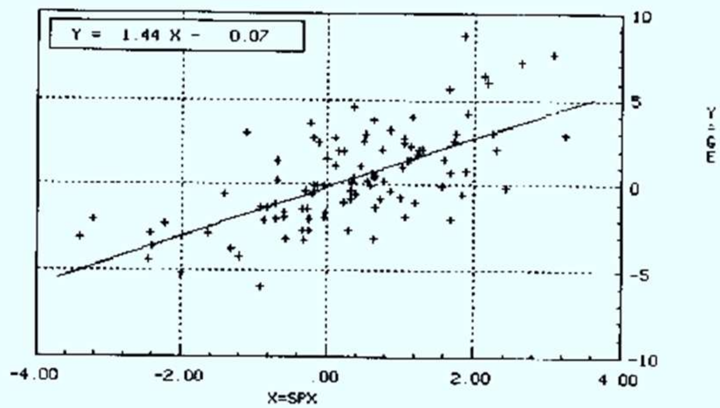
GE **US** GENERAL ELECTRIC CO.

Market SPX S&P 500 INDEX

Period W (D-W-M-Q-Y)
Range 2/18/94 To 2/ 9/96
T (T=Trade,B=Bid,A=Ask)

ADJ BETA	1.29
RAW BETA	1.44
Alpha (Intercept)	-.07
R2 (Correlation)	.44
Std Dev of Error	2.08
Std Error of Beta	.16
Number of Points	103

Adj beta = (0.67) * Raw Beta
+ (0.33) * 1.0



Example-GE

- You choose the S&P500 as your market proxy. You analyze the stock of General Electric (GE), and find (see later in the notes) that, using weekly returns, a $\alpha_j = -0.07\%,\ \beta_j = 1.44$

If you expect the S&P500 to increase to 5% next week, then according to the SIM, you expect the return on GE next week to be:

$$E[r_{GE}] = \alpha_{GE} + \beta_{GE} E[r_M] = -0.07\% + 1.44 \times 5\% = 7.13\%$$

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Example-GE

The variance of GE is $\sigma_{GE}^2 = \beta_{GE}^2\,\sigma_M^2 + \sigma^2[e_{GE}]$

Where $\sigma[e_{GE}]$ is the "Std. Dev. Of Error."

Over the sample, $\sigma_M = 1.28\%$ (from other data):

$$\sigma_{GE}^2 = \beta_{GE}^2\,\sigma_M^2 + \sigma^2[e_{GE}]$$

$$= (1.44)^2\,(1.28)^2 + (2.08)^2 = 3.40 + 4.33 = 7.73$$

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Summary

$$r_{GE} = \underbrace{\alpha_{GE}}_{\text{constant}} + \underbrace{\beta_{GE} r_M}_{\substack{\text{market-driven,} \\ \text{systematic}}} + \underbrace{e_{GE}}_{\substack{\text{non-market,} \\ \text{firm-specific}}}$$

$$\underbrace{\sigma^2_{GE}}_{\text{Total "risk"}} = \underbrace{\beta^2_{GE}\,\sigma^2_M}_{\substack{\text{Market or} \\ \text{systematic risk}}} + \underbrace{\sigma^2(e_{GE})}_{\substack{\text{Non-market,} \\ \text{firm-specific risk}}}$$

$$\underbrace{7.73}_{\text{Total "risk"}} = \underbrace{3.40}_{\substack{\text{Market or} \\ \text{systematic risk}}} + \underbrace{4.33}_{\substack{\text{Non-market,} \\ \text{firm-specific risk}}}$$

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Example-MSFT

MSFT Equity BETA

DG21 Equity **BETA**

HISTORICAL BETA

MSFT US

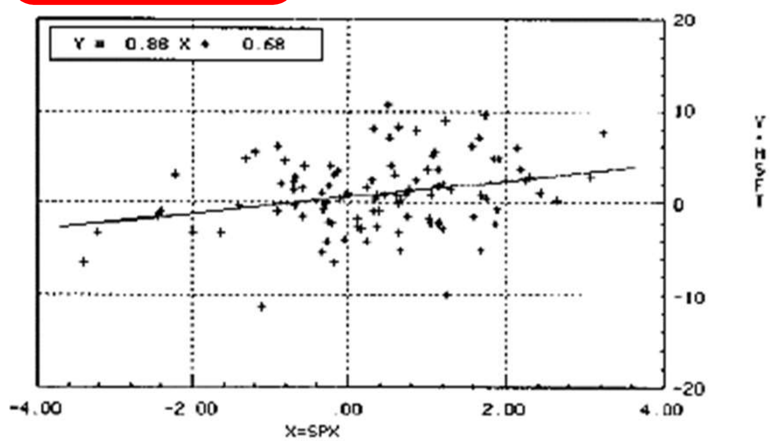
MICROSOFT CORP

Market SPX

S&P 500 INDEX

Period W (D-W-M-Q-Y)
Range 2/18/94 To 2/ 9/96
T (T=Trade,B=Bid,A=Ask)

ADJ BETA	.92
RAW BETA	.88
Alpha (Intercept)	.68
R2 (Correlation)	.08
Std Dev of Error	3.81
Std Error of Beta	.30
Number of Points	103



Adj beta = (0.67) * Raw Beta
+ (0.33) * 1.0

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Example-MSFT

Over the sample, $\sigma_M = 1.28\%$ (from other data)

$$\sigma^2_{MSFT} = \beta^2_{MSFT}\sigma^2_M + \sigma^2(e_{MSFT})$$

$$= (0.88)^2(1.28)^2 + (3.81)^2$$

$$= 1.27 + 14.52 = 15.79$$

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SIM and the Covariance between GE& MSFT

$$r_{GE} = \alpha_{GE} + \beta_{GE} r_M + e_{GE}$$

constant

firm-specific

$$r_{MSFT} = \alpha_{MSFT} + \beta_{MSFT} r_M + e_{MSFT}$$

The *only common influence* driving GE and MSFT is the market return r_M.

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Example

$$Cov\left(r_{GE}, r_{MSFT}\right) = \beta_{GE}\beta_{MSFT}\sigma_M^2 = (.88)(1.44)(1.28)^2 = 2.08$$

$$Corr\left(r_{GE}, r_{MSFT}\right) = 2.08 / \sqrt{7.33 \times 15.79} = 0.19$$

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Portfolio Risk & Return using SIM

$$E(R_p) = \alpha_p + \beta_p E(R_m)$$

$$\alpha_p = \sum_{i=1}^{N} w_i \alpha_i$$

$$\beta_p = \sum_{i=1}^{N} w_i \beta_i$$

w_i : The value of weight of security i in the portfolio

α_p : The alpha of a portfolio of securities

β_p : The beta of a portfolio of securities

N: The number of observations in the portfolio

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Portfolio Risk & Return using SIM

systematic risk

Total Risk of the Portfolio: Market Risk + Specific Portfolio Risk

unsystematic risk

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} w_i^2 \sigma_e^2$$

σ_p^2 : Variance of the portfolio

β_p^2 : The square of the portfolio beta

σ_m^2 :Variance of the market

w_i^2 : Square of the weight of security i

σ_{ei}^2 : Variance of the error of security i

Example

- Assume we formed a portfolio of GE and MSFT with 30%, 70% respectively, What is the expected return and the standard deviation of this portfolic

Security	Weight	Alpha	Beta
GE	0.30	-0.07	1.44
MSFT	0.70	0.68	0.88
Portfolio		0.455	1.048

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Portfolio Standard Deviation

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} w_i^2 \sigma_e^2$$

$= (1.048^2 * 1.28^2 + 11.46) = 13.26$

Security	Weight	Variance of the Error
GE	0.30	2.08^2
MSFT	0.70	3.81^2
Portfolio		11.46

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Questions & Answers

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