4-9 Erlang and Gamma Distributions

Erlang Distribution

The random variable X that equals the interval length until r counts occur in a Poisson process with mean $\lambda > 0$ has and **Erlang random variable** with parameters λ and r. The probability density function of X is

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}$$

for $x > 0$ and $r = 1, 2, 3, \ldots$.

4-9 Erlang and Gamma Distributions

Gamma Distribution

The **gamma function** is

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\, dx, \quad \text{for } r > 0 \qquad (4\text{-}17)$$

4-9 Erlang and Gamma Distributions

Gamma Distribution

The random variable X with probability density function

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, \quad \text{for } x > 0 \qquad (4\text{-}18)$$

has a **gamma random variable** with parameters $\lambda > 0$ and $r > 0$. If r is an integer, X has an Erlang distribution.

4-9 Erlang and Gamma Distributions

Gamma Distribution

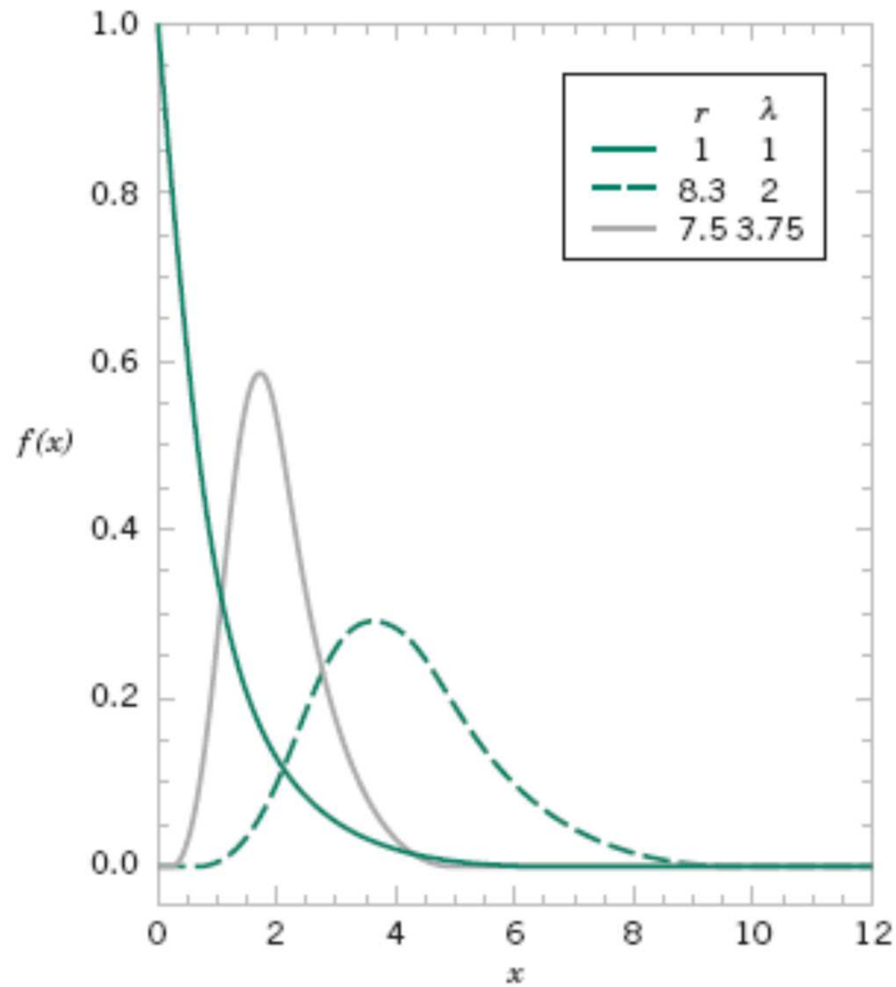

Figure 4-25 Gamma probability density functions for selected values of *r* and λ.

4-9 Erlang and Gamma Distributions

Gamma Distribution

If X is a **gamma random variable** with parameters λ and r,

$$\mu = E(X) = r/\lambda \quad \text{and} \quad \sigma^2 = V(X) = r/\lambda^2 \tag{4-19}$$

4-10 Weibull Distribution

Definition

The random variable X with probability density function

$$f(x) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right], \qquad \text{for } x > 0 \qquad (4\text{-}20)$$

is a **Weibull random variable** with scale parameter $\delta > 0$ and shape parameter $\beta > 0$.

4-10 Weibull Distribution

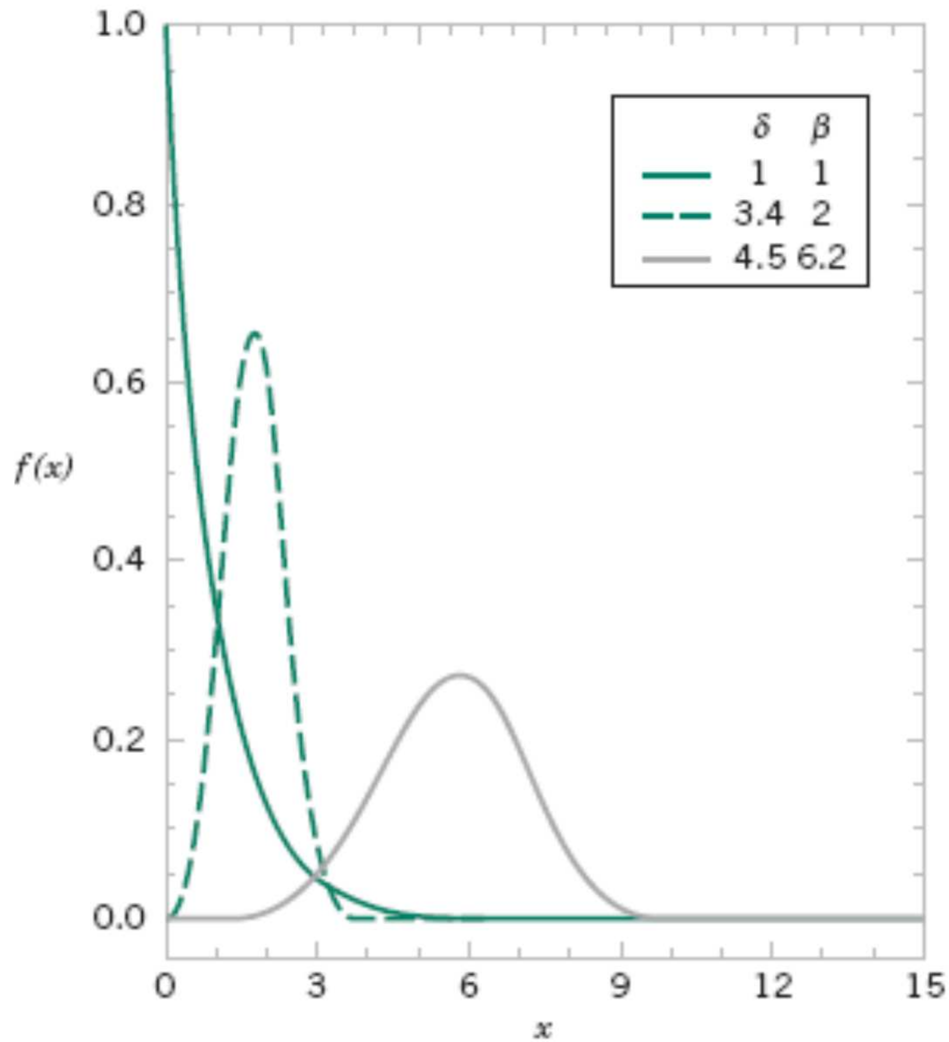

Figure 4-26 Weibull probability density functions for selected values of α and β.

4-10 Weibull Distribution

If X has a Weibull distribution with parameters δ and β, then the cumulative distribution function of X is

$$F(x) = 1 - e^{-\left(\frac{x}{\delta}\right)^{\beta}} \qquad (4\text{-}21)$$

If X has a Weibull distribution with parameters δ and β,

$$\mu = E(X) = \delta\Gamma\left(1 + \frac{1}{\beta}\right) \quad \text{and} \quad \sigma^2 = V(X) = \delta^2\Gamma\left(1 + \frac{2}{\beta}\right) - \delta^2\left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \qquad (4\text{-}21)$$

4-10 Weibull Distribution

Example 4-25

The time to failure (in hours) of a bearing in a mechanical shaft is satisfactorily mode[l]
Weibull random variable with $\beta = 1/2$, and $\delta = 5000$ hours. Determine the mean tin[e]
failure.

From the expression for the mean,

$$E(X) = 5000\Gamma[1 + (1/0.5)] = 5000\Gamma[3] = 5000 \times 2! = 10{,}000 \text{ hours}$$

Determine the probability that a bearing lasts at least 6000 hours. Now

$$P(x > 6000) = 1 - F(6000) = \exp -\left[\left(\frac{6000}{5000}\right)^{1/2}\right] = e^{-1.095} = 0.334$$

Consequently, only 33.4% of all bearings last at least 6000 hours.

4-11 Lognormal Distribution

Let W have a normal distribution mean θ and variance ω^2; then $X = \exp(W)$ is a **log-normal random variable** with probability density function

$$f(x) = \frac{1}{x\omega\sqrt{2\pi}} \exp\left[-\frac{(\ln x - \theta)^2}{2\omega^2}\right] \qquad 0 < x < \infty$$

The mean and variance of X are

$$E(X) = e^{\theta + \omega^2/2} \qquad \text{and} \qquad V(X) = e^{2\theta + \omega^2}(e^{\omega^2} - 1) \qquad (4\text{-}22)$$

The parameters of a lognormal distribution are θ and ω^2, but care is needed to interpret that these are the mean and variance of the normal random variable W. The mean and variance of X are the functions of these parameters shown in (4-22). Figure 4-27 illustrates lognormal distributions for selected values of the parameters.

4-11 Lognormal Distribution

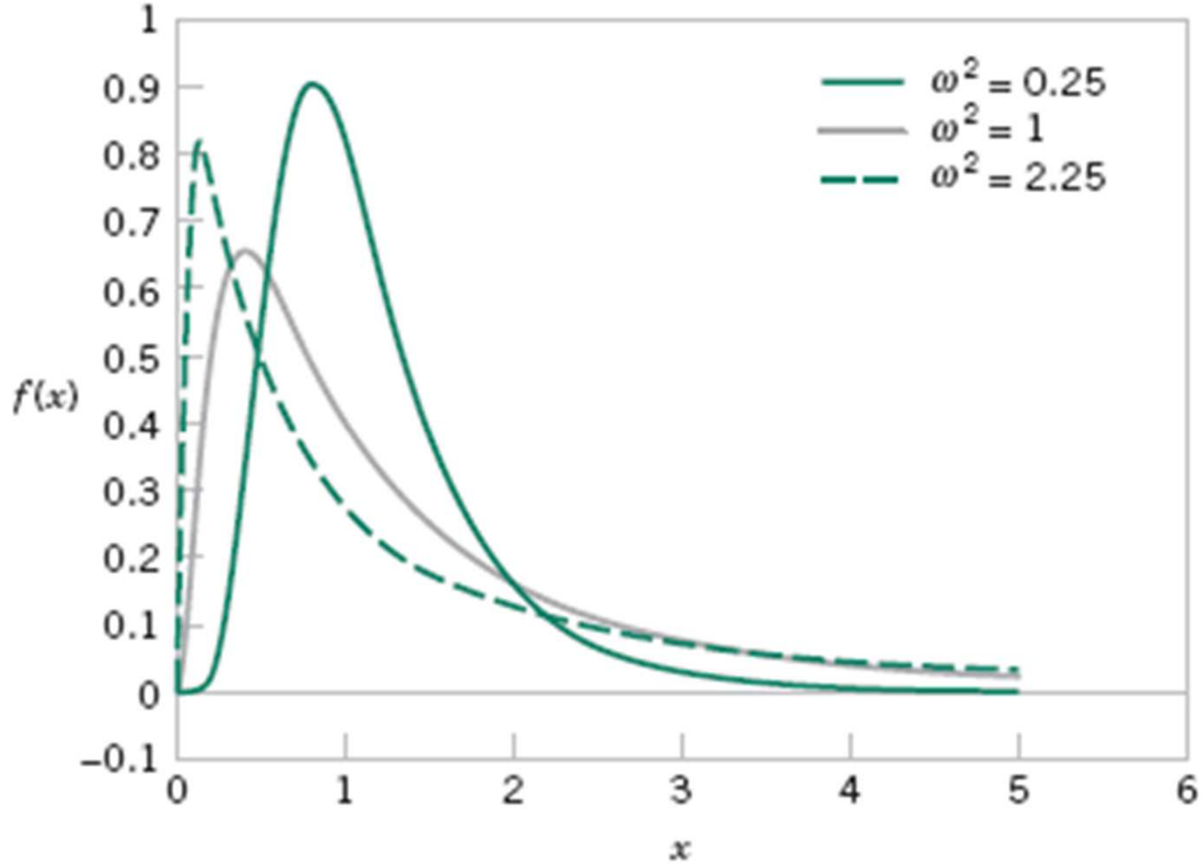

Figure 4-27 Lognormal probability density functions with $\theta = 0$ for selected values of ω^2.

4-11 Lognormal Distribution

Example 4-26

The lifetime of a semiconductor laser has a lognormal distribution with $\theta = 10$ hours and $\omega = 1.5$ hours. What is the probability the lifetime exceeds 10,000 hours?

From the cumulative distribution function for X

$$P(X > 10{,}000) = 1 - P[\exp(W) \le 10{,}000] = 1 - P[W \le \ln(10{,}000)]$$

$$= \Phi\left(\frac{\ln(10{,}000) - 10}{1.5}\right) = 1 - \Phi(-0.52) = 1 - 0.30 = 0.70$$

4-11 Lognormal Distribution

Example 4-26 (continued)

What lifetime is exceeded by 99% of lasers? The question is to determine x such that $P(X > x) = 0.99$. Therefore,

$$P(X > x) = P[\exp(W) > x] = P[W > \ln(x)] = 1 - \Phi\left(\frac{\ln(x) - 10}{1.5}\right) = 0.99$$

From Appendix Table II, $1 - \Phi(z) = 0.99$ when $z = -2.33$. Therefore,

$$\frac{\ln(x) - 10}{1.5} = -2.33 \quad \text{and} \quad x = \exp(6.505) = 668.48 \text{ hours.}$$

4-11 Lognormal Distribution

Example 4-26 (continued)

Determine the mean and standard deviation of lifetime. Now,

$$E(X) = e^{\theta + \omega^2/2} = \exp(10 + 1.125) = 67{,}846.3$$

$$V(X) = e^{2\theta + \omega^2}(e^{\omega^2} - 1) = \exp(20 + 2.25)[\exp(2.25) - 1] = 39{,}070{,}059{,}886.6$$

so the standard deviation of X is 197,661.5 hours. Notice that the standard deviation of lifetime is large relative to the mean.