4 Continuous Random Variables and Probability Distributions

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter you should be able to do the following:

1. Determine probabilities from probability density functions.
2. Determine probabilities from cumulative distribution functions and cumulative distribution functions from probability density functions, and the reverse.
3. Calculate means and variances for continuous random variables.
4. Understand the assumptions for each of the continuous probability distributions presented.
5. Select an appropriate continuous probability distribution to calculate probabilities in specific applications.
6. Calculate probabilities, determine means and variances for each of the continuous probability distributions presented.
7. Standardize normal random variables.
8. Use the table for the cumulative distribution function of a standard normal distribution to calculate probabilities.
9. Approximate probabilities for some binomial and Poisson distributions.

4-1 Continuous Random Variables

Previously, we discussed the measurement of the current in a thin copper wire. We noted that the results might differ slightly in day-to-day replications because of small variations in variables that are not controlled in our experiment—changes in ambient temperatures, small impurities in the chemical composition of the wire, current source drifts, and so forth.

Another example is the selection of one part from a day's production and very accurately measuring a dimensional length. In practice, there can be small variations in the actual measured lengths due to many causes, such as vibrations, temperature fluctuations, operator differences, calibrations, cutting tool wear, bearing wear, and raw material changes. Even the measurement procedure can produce variations in the final results.

4-2 Probability Distributions and Probability Density Functions

Figure 4-1 Density function of a loading on a long, thin beam.

4-2 Probability Distributions and Probability Density Functions

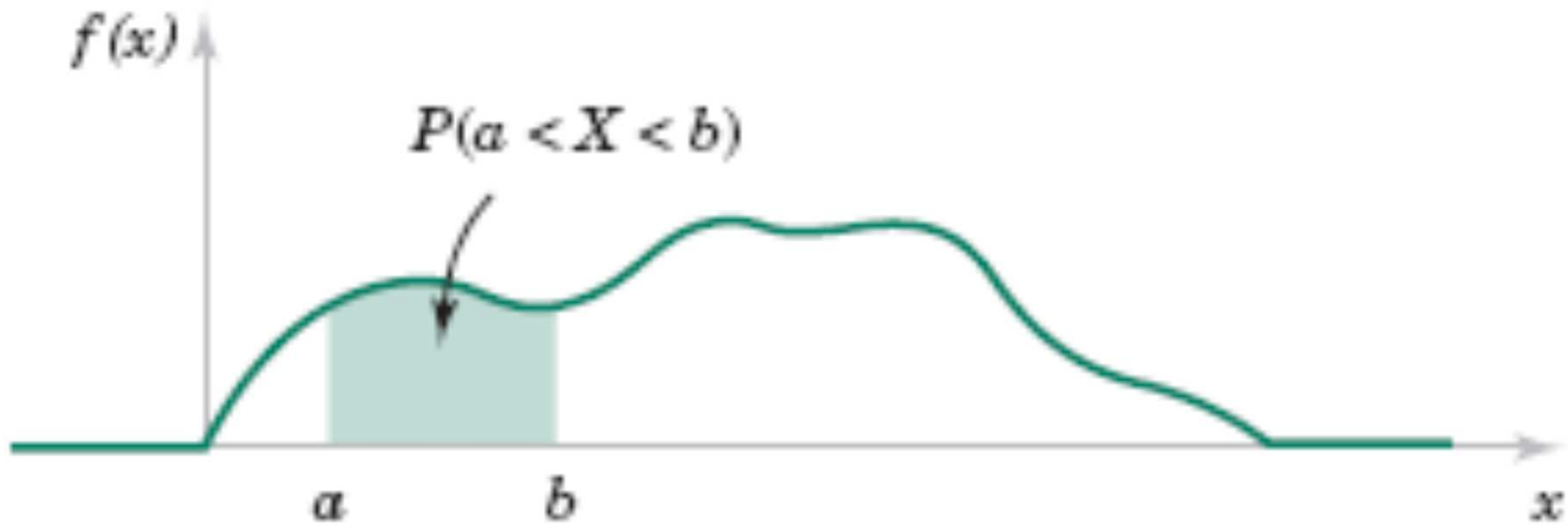

Figure 4-2 Probability determined from the area under $f(x)$.

4-2 Probability Distributions and Probability Density Functions

Definition

For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

(3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx =$ area under $f(x)$ from a to b

for any a and b (4-1)

4-2 Probability Distributions and Probability Density Functions

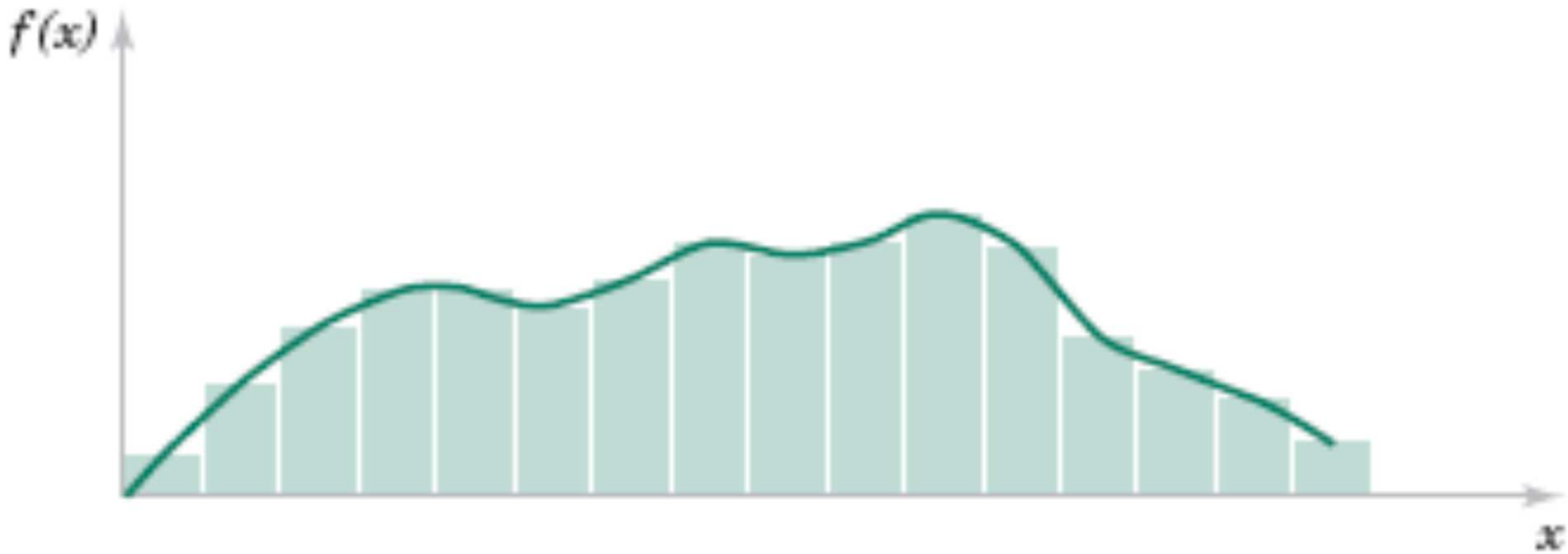

Figure 4-3 Histogram approximates a probability density function.

4-2 Probability Distributions and Probability Density Functions

If X is a **continuous random variable**, for any x_1 and x_2,

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2) \quad (4\text{-}2)$$

4-2 Probability Distributions and Probability Density Functions

Example 4-2

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function $f(x) = 20e^{-20(x-12.5)}$, $x \geq 12.5$.

If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scrapped? The density function and the requested probability are shown in Fig. 4-5. A part is scrapped if $X > 12.60$. Now,

$$P(X > 12.60) = \int_{12.6}^{\infty} f(x)\,dx = \int_{12.6}^{\infty} 20e^{-20(x-12.5)}\,dx = -e^{-20(x-12.5)}\Big|_{12.6}^{\infty} = 0.135$$

4-2 Probability Distributions and Probability Density Functions

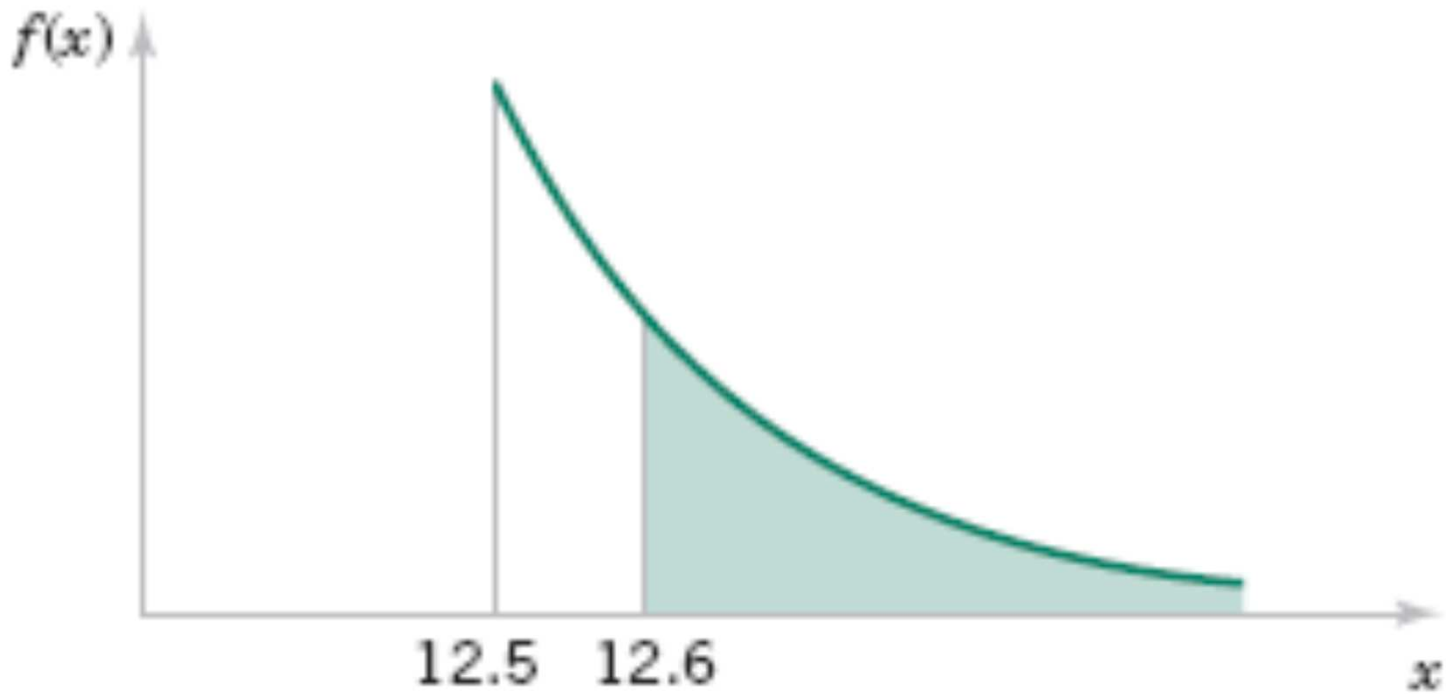

Figure 4-5 Probability density function for Example 4-2.

4-2 Probability Distributions and Probability Density Functions

Example 4-2 (continued)

What proportion of parts is between 12.5 and 12.6 millimeters? Now,

$$P(12.5 < X < 12.6) = \int_{12.5}^{12.6} f(x)\, dx = -e^{-20(x-12.5)} \Big|_{12.5}^{12.6} = 0.865$$

Because the total area under $f(x)$ equals 1, we can also calculate $P(12.5 < X < 12.6) = 1 - P(X > 12.6) = 1 - 0.135 = 0.865$.

4-3 Cumulative Distribution Functions

Definition

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\, du \qquad (4\text{-}3)$$

for $-\infty < x < \infty$.

4-3 Cumulative Distribution Functions

Example 4-4

For the drilling operation in Example 4-2, $F(x)$ consists of two expressions.

$$F(x) = 0 \quad \text{for} \quad x < 12.5$$

and for $12.5 \le x$

$$F(x) = \int_{12.5}^{x} 20e^{-20(u-12.5)}\, du$$
$$= 1 - e^{-20(x-12.5)}$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 12.5 \\ 1 - e^{-20(x-12.5)} & 12.5 \le x \end{cases}$$

Figure 4-7 displays a graph of $F(x)$.

4-3 Cumulative Distribution Functions

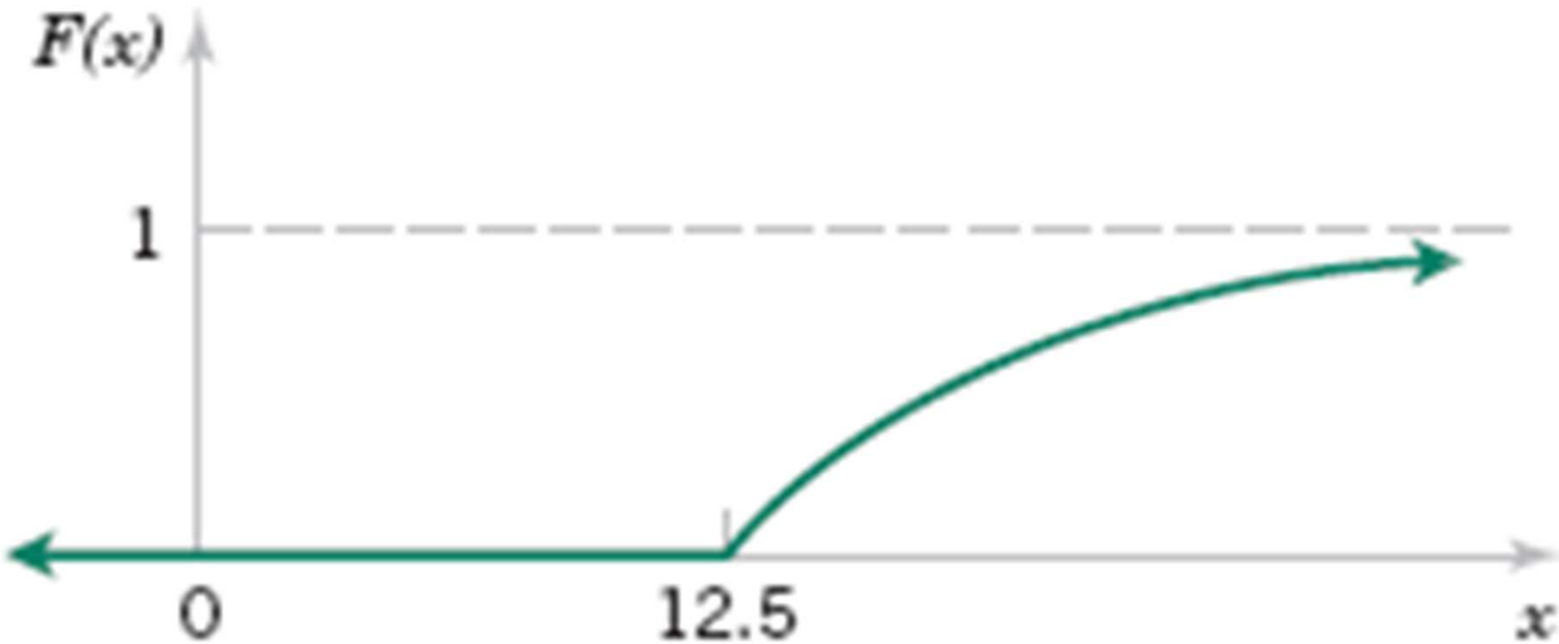

Figure 4-7 Cumulative distribution function for Example 4-4.

4-4 Mean and Variance of a Continuous Random Variable

Definition

Suppose X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \tag{4-4}$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

4-4 Mean and Variance of a Continuous Random Variable

Example 4-6

For the copper current measurement in Example 4-1, the mean of X is

$$E(X) = \int_0^{20} xf(x)\,dx = 0.05x^2/2 \,\Big|_0^{20} = 10$$

The variance of X is

$$V(X) = \int_0^{20} (x-10)^2 f(x)\,dx = 0.05(x-10)^3/3 \,\Big|_0^{20} = 33.33$$

4-4 Mean and Variance of a Continuous Random Variable

Expected Value of a Function of a Continuous Random Variable

If X is a continuous random variable with probability density function $f(x)$,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x)\,dx \qquad (4\text{-}5)$$

4-4 Mean and Variance of a Continuous Random Variable

Example 4-8

For the drilling operation in Example 4-2, the mean of X is

$$E(X) = \int_{12.5}^{\infty} x f(x)\,dx = \int_{12.5}^{\infty} x\,20e^{-20(x-12.5)}\,dx$$

Integration by parts can be used to show that

$$E(X) = -xe^{-20(x-12.5)} - \frac{e^{-20(x-12.5)}}{20}\Bigg|_{12.5}^{\infty} = 12.5 + 0.05 = 12.55$$

The variance of X is

$$V(X) = \int_{12.5}^{\infty} (x - 12.55)^2 f(x)\,dx$$

Although more difficult, integration by parts can be used two times to show that $V(X) = 0.0025$.

4-5 Continuous Uniform Random Variable

Definition

A continuous random variable X with probability density function

$$f(x) = 1/(b - a), \qquad a \le x \le b \tag{4-6}$$

is a **continuous uniform random variable.**

4-5 Continuous Uniform Random Variable

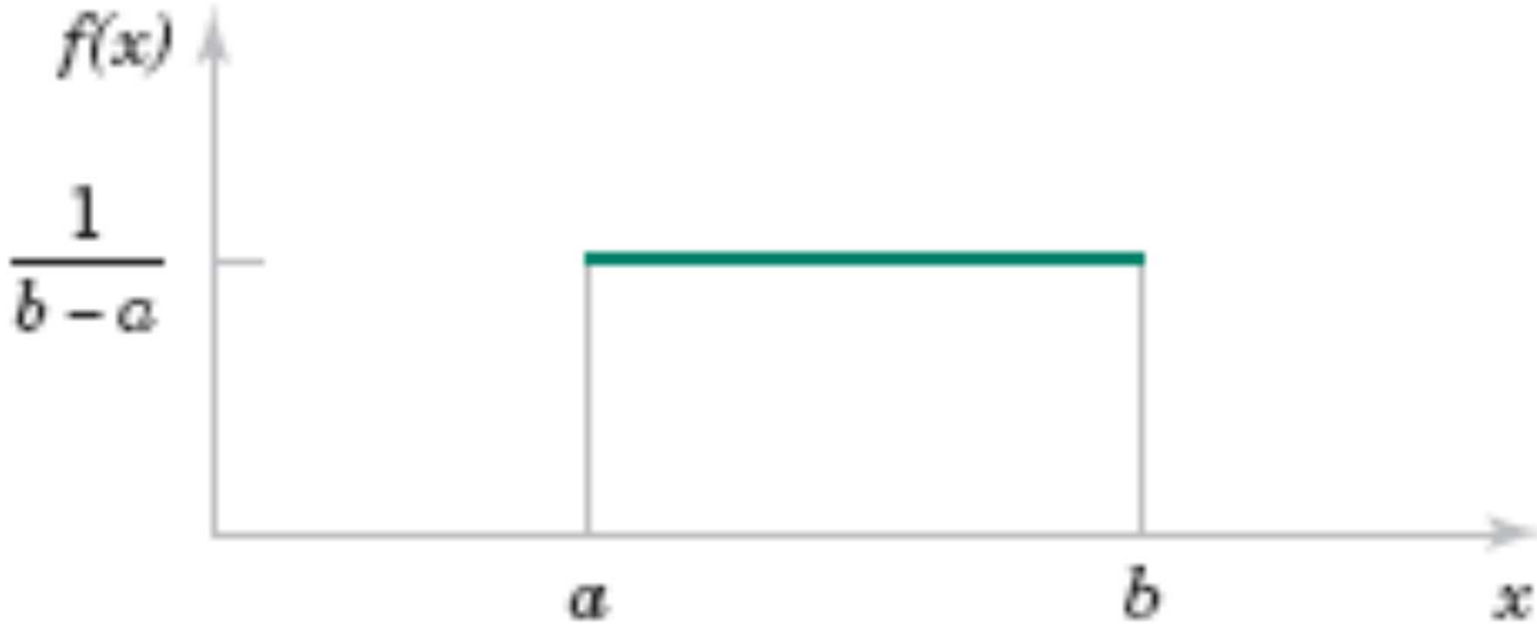

Figure 4-8 Continuous uniform probability density function.

4-5 Continuous Uniform Random Variable

Mean and Variance

If X is a continuous uniform random variable over $a \le x \le b$,

$$\mu = E(X) = \frac{(a+b)}{2} \quad \text{and} \quad \sigma^2 = V(X) = \frac{(b-a)^2}{12} \tag{4-7}$$

4-5 Continuous Uniform Random Variable

Example 4-9

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is [0, 20 mA], and assume that the probability density function of X is $f(x) = 0.05, 0 \leq x \leq 20$.

What is the probability that a measurement of current is between 5 and 10 milliamperes? The requested probability is shown as the shaded area in Fig. 4-9.

$$P(5 < X < 10) = \int_{5}^{10} f(x)\, dx$$

$$= 5(0.05) = 0.25$$

The mean and variance formulas can be applied with $a = 0$ and $b = 20$. Therefore,

$$E(X) = 10 \text{ mA} \quad \text{and} \quad V(X) = 20^2/12 = 33.33 \text{ mA}^2$$

Consequently, the standard deviation of X is 5.77 mA.

4-5 Continuous Uniform Random Variable

Figure 4-9 Probability for Example 4-9.

4-5 Continuous Uniform Random Variable

The cumulative distribution function of a continuous uniform random variable is obtained by integration. If $a < x < b$,

$$F(x) = \int_a^x 1/(b-a)\,du = x/(b-a) - a/(b-a)$$

Therefore, the complete description of the cumulative distribution function of a continuous uniform random variable is

$$F(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \le x < b \\ 1 & b \le x \end{cases}$$