2-4 Conditional Probability

- To introduce conditional probability, consider an example involving manufactured parts.
- Let D denote the event that a part is defective and let F denote the event that a part has a surface flaw.
- Then, we denote the probability of D given, or assuming, that a part has a surface flaw as $P(D|F)$. This notation is read as the **conditional probability** of D given F, and it is interpreted as the probability that a part is defective, given that the part has a surface flaw.

2-4 Conditional Probability

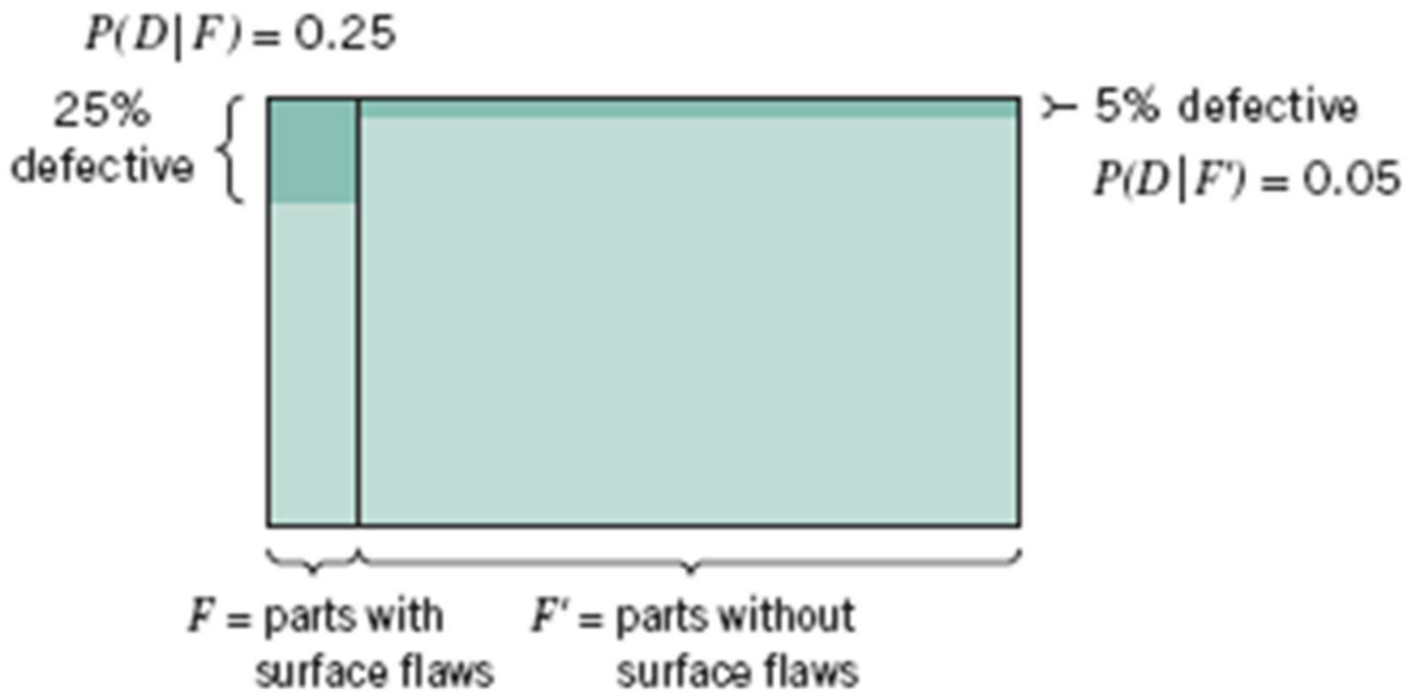


Figure 2-13 Conditional probabilities for parts with surface flaws

2-4 Conditional Probability

Definition

The **conditional probability** of an event B given an event A, denoted as $P(B|A)$, is

$$P(B|A) = P(A \cap B)/P(A) \qquad (2\text{-}9)$$

for $P(A) > 0$.

2-5 Multiplication and Total Probability Rules

2-5.1 Multiplication Rule

$$P(A \cap B) = P(B|A)P(A) = P(A|B)P(B) \qquad (2\text{-}10)$$

2-5 Multiplication and Total Probability Rules

Example 2-26

The probability that the first stage of a numerically controlled machining operation for high-rpm pistons meet specifications is 0.90. Failures are due to metal variations, fixture alignment, cutting blade condition, vibration, and ambient environmental conditions. Given that the first stage meets specifications the probability that a second stage of machining meets specifications is 0.95. What is the probability that both stages meet specifications?

Let A and B denote the events that the first and second stages meet specifications, respectively. The probability requested is

$$P(A \cap B) = P(B|A)P(A) = 0.95(0.90) = 0.855$$

Although it is also true that $P(A \cap B) = P(A|B)P(B)$, the information provided in the problem does not match this second formulation.

2-5 Multiplication and Total Probability Rules

2-5.2 Total Probability Rule

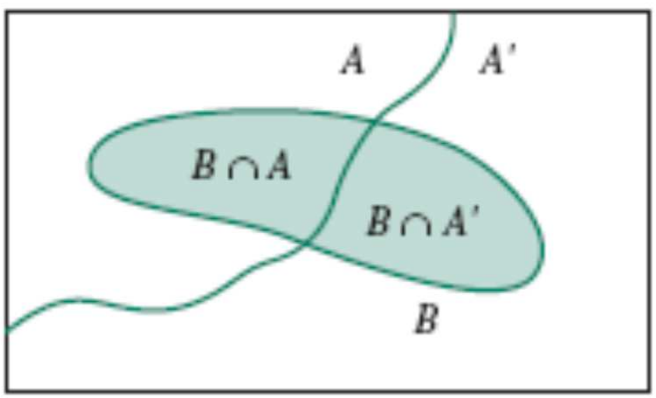
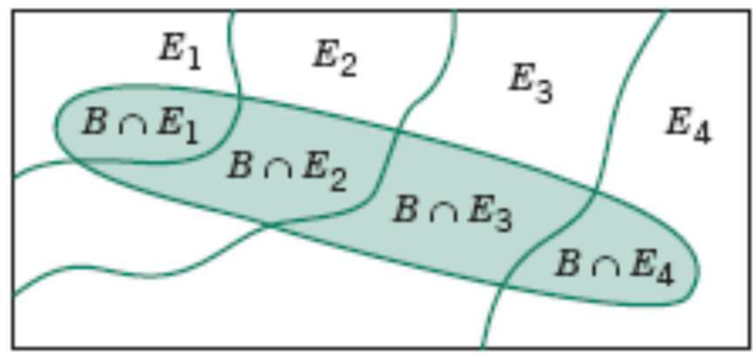


Figure 2-15 Partitioning an event into two mutually exclusive subsets.



$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$

Figure 2-16 Partitioning an event into several mutually exclusive subsets.

2-5 Multiplication and Total Probability Rules

2-5.2 Total Probability Rule (two events)

For any events A and B,

$$P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A') \quad (2\text{-}11)$$

2-5 Multiplication and Total Probability Rules

Example 2-27

Consider the contamination discussion at the start of this section. The information is summarized here

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not High	0.8

Let F denote the event that the product fails, and let H denote the event that the chip is exposed to high levels of contamination. The requested probability is $P(F)$, and the information provided can be represented as

$$P(F \mid H) = 0.10 \quad \text{and} \quad P(F \mid H') = 0.005$$
$$P(H) = 0.20 \quad \text{and} \quad P(H') = 0.80$$

From Equation 2-11,

$$P(F) = 0.10(0.20) + 0.005(0.80) = 0.024$$

which can be interpreted as just the weighted average of the two probabilities of failure.

2-5 Multiplication and Total Probability Rules

Total Probability Rule (multiple events)

Assume $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive sets. Then

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\ &= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k) \qquad (2\text{-}12) \end{aligned}$$

2-6 Independence

Definition (two events)

Two events are **independent** if any one of the following equivalent statements is true:

(1) $P(A|B) = P(A)$

(2) $P(B|A) = P(B)$

(3) $P(A \cap B) = P(A)P(B)$ (2-13)

2-6 Independence

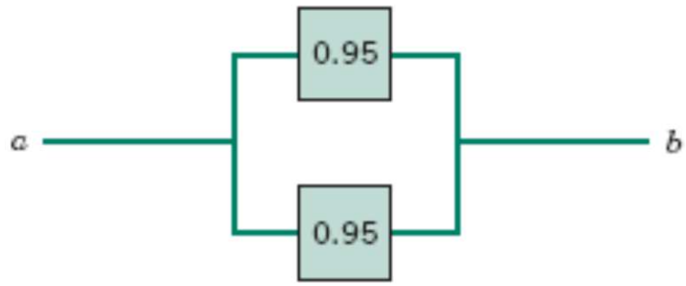
Definition (multiple events)

The events $E_1, E_2, \ldots, E_n$ are independent if and only if for any subset of these events $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$,

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \times P(E_{i_2}) \times \cdots \times P(E_{i_k}) \qquad (2\text{-}14)$$

Example 2-34

The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?



Let T and B denote the events that the top and bottom devices operate, respectively. There is a path if at least one device operates. The probability that the circuit operates is

$$P(T \text{ or } B) = 1 - P[(T \text{ or } B)'] = 1 - P(T' \text{ and } B')$$

a simple formula for the solution can be derived from the complements T' and B'. From the independence assumption,

$$P(T' \text{ and } B') = P(T')P(B') = (1 - 0.95)^2 = 0.05^2$$

so

$$P(T \text{ or } B) = 1 - 0.05^2 = 0.9975$$

2-7 Bayes' Theorem

Definition

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad \text{for} \quad P(B) > 0 \qquad (2\text{-}15)$$

2-7 Bayes' Theorem

Bayes' Theorem

If $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1|B) = \frac{P(B|E_1)P(E_1)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k)} \quad (2\text{-}16)$$

for $P(B) > 0$

Example 2-37

Because a new medical procedure has been shown to be effective in the early detection of an illness, a medical screening of the population is proposed. The probability that the test correctly identifies someone with the illness as positive is 0.99, and the probability that the test correctly identifies someone without the illness as negative is 0.95. The incidence of the illness in the general population is 0.0001. You take the test, and the result is positive. What is the probability that you have the illness?

Let D denote the event that you have the illness, and let S denote the event that the test signals positive. The probability requested can be denoted as $P(D|S)$. The probability that the test correctly signals someone without the illness as negative is 0.95. Consequently, the probability of a positive test without the illness is

$$P(S|D') = 0.05$$

From Bayes' Theorem,

$$\begin{aligned} P(D|S) &= P(S|D)P(D)/[P(S|D)P(D) + P(S|D')P(D')] \\ &= 0.99(0.0001)/[0.99(0.0001) + 0.05(1 - 0.0001)] \\ &= 1/506 = 0.002 \end{aligned}$$

That is, the probability of now having the illness given a positive result from the test is only 0.002. Surprisingly, even though the test is effective, in the sense that $P(S|D)$ is high and $P(S|D')$ is low, because the incidence of the illness in the general population is low, the chances are quite small that you actually have the disease even if the test is positive.

2-8 Random Variables

Definition

A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.

A random variable is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as $x = 70$ milliamperes.

2-8 Random Variables

Definition

A **discrete** random variable is a random variable with a finite (or countably infinite) range.
A **continuous** random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

2-8 Random Variables

Examples of Random Variables

Examples of **continuous** random variables:

electrical current, length, pressure, temperature, time, voltage, weight

Examples of **discrete** random variables:

number of scratches on a surface, proportion of defective parts among 1000 tested, number of transmitted bits received in error

IMPORTANT TERMS AND CONCEPTS

Addition rule
Axioms of probability
Bayes' theorem
Combination
Conditional probability
Equally likely outcomes
Event
Independence
Multiplication rule
Mutually exclusive events
Outcome
Permutation
Probability
Random experiment
Random variables—discrete and continuous
Sample spaces—discrete and continuous
Simpson's paradox
Total probability rule
Tree Diagram
Venn Diagram
With or without replacement